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BARTLETT'S
SPHERICAL ASTRONOMY.

ELEMENTS

OF

NATURAL PHILOSOPHY

BY

W. H. C. BARTLETT, LL.D.,

PROFESSOR OF NATURAL AND EXPERIMENTAL PHILOSOPHY IN THE UNITED STATES
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AUTHOR OF
"ELEMENTS OF MECHANICS," "ACOUSTICS," "OPTICS," AND
"ANALYTICAL MECHANICS."

IV.—SPHERICAL ASTRONOMY.

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PREFACE.

The work here offered to the public was undertaken by its author to supply a want long felt in his own department of instruction in the Military Academy at West Point. Its aim is to present a concise course of Spherical Astronomy in its relationship to Celestial Mechanics, of which it is the offspring. The solar and stellar systems are, therefore, assumed and described as necessary facts, arising from the detached condition of the bodies which compose them and the laws of universal gravitation. The consequences from these systems, to a spectator on the earth, are then deduced, and their entire coincidence with the celestial phenomena, as they arise spontaneously, is relied upon as full and sufficient justification for the assumption, and as proof that the systems are true. This forms the first part of the subject. A general account of the methods by which the future condition and aspects of the heavens are predicted follows, and the more important applications to the current wants of Navigation, Geography, and Chronology, conclude the volume.

In the description and discussions of instruments, those only have been selected which are best suited to convey a full view of the whole theory and practice of Astronomical Measurements.

The author would acknowledge his obligation to Sir John Herschel, Professor Challis, Mr. Maddy, Mr. Francis Bailey Mr. De Morgan, Mr. Woolhouse, M. Francœur, M. De Launay and M. Briot, whose works have been constantly before him.

CONTENTS.

TABLES.

The Greek Alphabet is here inserted to aid those who are not already familiar with it in reading the parts of the text in which its letters occur:

Letters.	Names.	Letters.	Names.
Α α	Alpha	Ν ν	Nu
Β β ϐ	Bēta	Ξ ξ	Xi
Γ γ	Gamma	Ο ο	Omicron
Δ δ	Delta	Π ϖ π	Pi
Ε ε	Epsilon	Ρ ρ ϱ	Rho
Ζ ζ	Zēta	Σ σ ς	Sigma
Η η	Eta	Τ τ	Tau
Θ ϑ θ	Thēta	Υ υ	Upsilon
Ι ι	Iōta	Φ φ	Phi
Κ κ	Kappa	Χ χ	Chi
Λ λ	Lambda	Ψ ψ	Psi
Μ μ	Mu	Ω ω	Omega

Conventional Signs used in Astronomy.

L, for mean longitude,
M, — mean anomaly,
V, — true anomaly,
μ, — mean daily sidereal motion,
r, — radius vector,
φ, — angle of eccentricity,
π, — longitude of perihelion,
α, — right ascension,
δ, — declination,
Δ, — logarithm of distance from the earth,
l, — heliocentric longitude,
b, — heliocentric latitude,
λ, — geocentric longitude,
β, — geocentric latitude,
☊, — longitude of ascending node,
i, — inclination of orbit to the ecliptic,
w, — angular distance from perihelion to node,
u, — distance from node, or argument for latitude.

ASTRONOMY.

ASTRONOMY.

§ 1. The science which treats of the heavenly bodies is called *Astronomy.* It is divided into *Physical* and *Spherical* Astronomy.

§ 2. Physical Astronomy is a system of Mechanics, in which the forces are *universal gravitation* and *inertia*, and the objects the gigantic masses that move through indefinite space. It treats of the physical conditions of the heavenly bodies, their mutual actions on each other, and explains the causes of the celestial phenomena.

§ 3. Spherical Astronomy is mainly concerned with the appearances, magnitudes, motions, arrangements, and distances of the heavenly bodies; and seeks to apply the deductions from these to the practical wants of society. It is a science of observation, and its principal means of investigation are Optical and Mathematical Instruments. This branch of Astronomy will form the subject of the present volume.

§ 4. No subject calls more strongly upon the student to abandon first impressions than Astronomy. All its conclusions are in striking contradiction to those of superficial observation, and to what appears, at first view, the most positive evidence of the senses.

§ 5. Every student approaches it for the first time with a firm belief that he lives on something fixed, and, abating the inequalities of hill and valley, that this something is a flat surface of indefinite extent, composed of land and water; and that the blue firmament which he sees around and above him in the distance is a stationary vault, upon the surface of which appear to be placed all objects out of contact with the ground.

§ 6. The Earth on which he stands is divested by Astronomy of its flattened shape and of its character of fixidity, and is shown to be a globular body turning swiftly about its centre, and moving onward through space with great rapidity. It teaches him that his vault has no existence

in fact, and is but a mere illusion which comes from looking through the indefinite space, extended without limit, in which he is moving.

§ 7. Were the Earth reduced to a mere point, and a spectator placed upon it, he would see around him at one view all the bodies which make up the visible universe; and in the absence of any means of judging of their distances from him, would refer them in the direction in which they were seen from his station, to the concave surface of an imaginary sphere, having its centre at his eye and its surface at some vast and indefinite distance.

SOLAR SYSTEM.

§ 8. A little observation would lead him to conclude that by far the greater number of these bodies appear fixed while the rest seem ever on the move, continually shifting their positions with respect to those which appear fixed, and to each other. The former are called FIXED STARS: the latter compose what is called the SOLAR SYSTEM, a group of bodies from which the fixed stars are so remote as to produce upon it no appreciable influence.

§ 9. All bodies attract one another with intensities which are proportional to the quantity of the attracting masses directly, and to the squares of the distances inversely, **Analyt. Mech.**, § 205.

§ 10. Bodies resist by their inertia all change in their actual state of motion; this resistance is exerted simultaneously with the change, and is always equal in intensity, and contrary in direction, to the force which produces it.

§ 11. The bodies of the solar system have motions that carry them in directions oblique to the lines along which their mutual attractions are exerted. The attractive forces draw them aside from these directions; inertia resists by an equal and contrary reaction; and the bodies are forced into curvilinear paths, and made to revolve about the centre of inertia of the whole.

§ 12. Thus, the antagonistic forces of gravitation and of inertia are the simple but efficient causes which keep the bodies of the solar system together as a single group, and impress upon it a character of stability and perpetuity. But for the force of gravitation the bodies would separate more and more, and wander through endless space; and but for the force of inertia, that of gravitation would pile them together in one confused mass.

§ 13. The force of gravitation increases rapidly with a diminution, and decreases as rapidly with an augmentation, of distance. Those bodies which are nearest exert, therefore, the greatest influence upon one another's

motions. Bodies composing an insulated group may perform their evolutions among each other undisturbed by the action of those without, provided the distances of the latter be very great in comparison to those which separate the individuals of the group.

§ 14. This is a characteristic of the solar system. Its own dimensions, vast as they are when expressed in terms of any linear unit with which we are familiar, are utterly insignificant when compared with its distance from the fixed stars. Each of the latter, by virtue of this relatively great distance, acting upon all the bodies of the system equally and in parallel directions, the effect of the whole can only be to move the group collectively through space.

§ 15. The same thing takes place upon a smaller scale within the solar system itself. Some of its members are so close together, and at the same time so far removed from the others, as to be forced to revolve about one another, while the combined action of the rest carries them as a sub-group, so to speak, about the centre of inertia of the whole.

§ 16. The mass of the sun so far exceeds the sum of the masses of all the other bodies of the system, as to throw the centre of inertia of the whole group within the boundary of its own volume; and although the centre of the sun actually revolves about this point, yet its motion becomes so small, when viewed from the distance of the earth, that it is insensible except through the medium of the most refined instruments. All the other bodies are, therefore, said to revolve about the sun as a centre, and it is from this fact, and the controlling influence which this latter body exerts over the motions of all the others, that the system takes its name.

§ 17. The same is true of the sub-groups; the mass of one of the bodies in each being so much greater than the sum of the masses of the rest as to cause the latter to revolve approximately about its centre, while this centre revolves about the sun.

§ 18. The path a body describes about another as a principal source of attraction, is called an *orbit.*

§ 19. Those bodies which describe their orbits about the sun are called *primary*, and those which describe their orbits about the primaries are called *secondary bodies.* These latter are also called Satellites.

Of the primary bodies there are three distinct classes, differing from each other mainly in the shape of their orbits, their densities, and general aspects.

§ 20. A body subjected to the action of a central force, whose intensity varies as the square of the distance inversely, must describe one or other of

the conic sections, depending upon the relation between its velocity and the intensity of the central force. The orbits that are known to belong to the solar system are ellipses.

§ 21. Those primaries which move in elliptical orbits of small eccentricities are called PLANETS. Those primaries having orbits of great eccentricities are called COMETS. Comets are also distinguished from planets in having a degree of density so low as to give some the appearance more of a vapor than of a solid body.

§ 22. The solar system consists then of the *Sun*, *Planets*, *Comets*, and *Satellites.* Setting out from the sun, the known planets, with their names, occur in the following order, viz.: *Mercury, Venus,* the *Earth, Mars,* then a class called the *Planetoids*, of which ninety-one are known at the present time, *Jupiter*, *Saturn*, *Uranus,* and *Neptune.* See Plate I., Fig. 1.

To these must be added a multitude of much smaller bodies of the nature of planetoids, whose existence is inferred from the fact that some of their number make their way now and then to the earth's surface under the name of *meteors.*

§ 23. It would be utterly impossible to give within the narrow limits of an octavo page a graphical representation of the relative dimensions of the solar system ; and to aid the conceptions of the student, Sir John Herschel has instituted the following illustration, viz. : On any well-levelled field place a globe two feet in diameter ; this will represent the sun ; Mercury will be represented by a grain of mustard-seed on the circumference of a circle 164 feet in diameter for its orbit ; Venus a pea on the circumference of a circle 284 feet in diameter ; the Earth also a pea on the circumference of a circle 430 feet in diameter ; Mars a rather large pin's head on the circumference of a circle of 654 feet diameter ; the Planetoids grains of sand on circular orbits varying from 1000 to 1200 feet in diameter ; Jupiter a moderate sized orange on a circumference nearly half a mile in diameter ; Saturn a small orange on the circumference of a circle four-fifths of a mile in diameter ; Uranus a full sized cherry on the circumference of a circle more than a mile and a half in diameter ; and Neptune a good sized plum on the circumference of a circle about two miles and a half in diameter. To illustrate the relative motions, Mercury must describe a portion of its orbit equal in length to its own diameter in 41 seconds ; Venus in 4 minutes and 14 seconds ; the Earth in 7 minutes ; Mars in 4 minutes and 48 seconds ; Jupiter in 2 hours and 56 minutes ; Saturn in 3 hours and 13 minutes ; Uranus in 2 hours and 16 minutes, and Neptune in 3 hours and 30 minutes. Now conceive the two feet globe to be increased till its diameter becomes 880,000 English miles, and suppose the

Plate I.

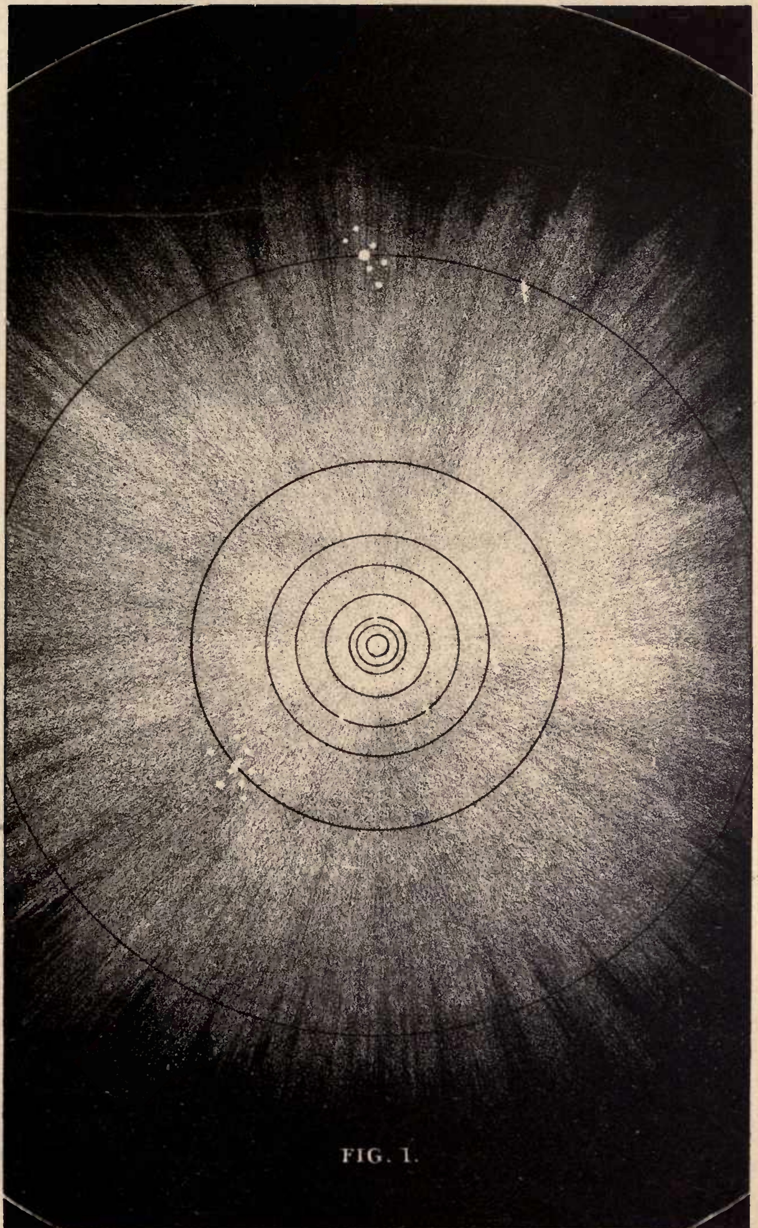


FIG. 1.

TO FRONT PAGE 4.

other bodies and their distances increased in the same proportion; the result will represent the dimensions of the solar system. It will give to the earth a diameter of nearly eight thousand miles, a distance from the sun equal to 95 millions of miles, and a velocity through space, around the sun, of 19 miles a second.

The orbits, although referred to as circles, are in fact ellipses, but of eccentricities so small as to justify the substitution for the mere purposes of the illustration.

§ 24. The fixed stars are self-luminous. The sun is regarded as one of this class of bodies, and by its greater proximity to the earth, becomes the principal source of heat and light to its inhabitants.

§ 25. The planets and satellites are opaque non-luminous bodies, and are visible only in consequence of light received from the sun and reflected to the earth.

SPHERICAL ASTRONOMY.

MOTION.

§ 26. Motion signifies the condition of a body, in virtue of which it oc cupies successively different places. But we can form no idea of place except by referring it to other places, and these again, to be known, must be referred to others, and so without limit; so that place is, in its very nature, entirely relative. Motion is, from its definition, therefore, also relative.

§ 27. We judge of the rate of motion by the greater or less rapidity with which the object possessing it varies its distance from other objects assumed as origins. These origins may themselves be in motion, but if the circumstances of the spectator be such as to deceive him into the belief that they are at rest, he will attribute all change of distance to a motion wholly in the object which he refers to them. And this is one of the most fruitful sources of the many erroneous notions with which students generally commence the study of astronomy.

§ 28. If two objects be in motion, and they alone occupy the spectator's field of view, the effect to him will be the same if he suppose one fixed, and attribute the whole of its motion to the other in a contrary direction; for this will not alter the rate by which they approach to or recede from one another.

PARALLACTIC MOTION AND PARALLAX.

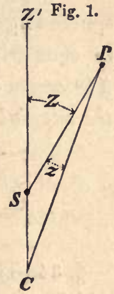
§ 29. The real motion of a spectator gives rise to the appearance of motion among surrounding objects which are relatively at rest. Objects in front of him seem to separate from one another, those behind appear to approach one another, and those directly to the right and left seem to move in a direction parallel to his own motion.

A spectator, for example, travelling over a plain studded with trees or other objects will, on fixing his eyes upon a single object without withdrawing his attention from the general landscape, see or think he sees the

latter in rotary motion about that object as a centre; all objects between it and himself appearing to move backward, or contrary to his own motion, and all beyond it, forward or in the direction in which he moves.

This apparent change in the relative places of objects, arising from a shifting of the point of view from which they are seen, is called *parallactic motion;* and the amount of angular change in the instance of any particular object is called the *parallax* of that object.

Fig. 1.



§ 30. Let P be the place of an object, C and S the places from which it is seen; and let its place be referred to some point Z', on the prolongation of the line CS, which joins the points of view. The angular change in the place of P as seen from C and S will be

$$Z'SP - Z'CP = SPC = \text{the parallax of } P.$$

That is to say, *the parallax of an object is the angle subtended at the object by the distance between the stations from which it is seen.*

Make $CP = d$; $CS = \rho$; the angle $Z'SP = Z$; the angle $SPC = z$ Then from the triangle CSP, we have

$$\sin z = \frac{\rho}{d} . \sin Z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Whence the parallax increases with an increase of the spectator's change of place, with diminution of the object's distance, and also with the approximation of Z to 90°.

§ 31. All other things being equal, the parallax will be less as the object's distance is greater; and when the parallax is zero for any arbitrary value of Z, the factor $\frac{\rho}{d}$ must be zero, and the change of the spectator's place must be utterly insignificant in comparison with the object's distance

CELESTIAL SPHERE.

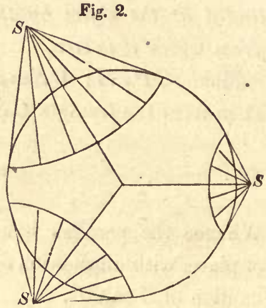
§ 32. Now, when the heavens are examined it is found that by far the greater number of the celestial bodies have no sensible parallax, while comparatively a few have. The first are the *fixed stars;* and they are so called from the fact that they always preserve the same angular distances from any assumed point and from each other, from whatever station on the earth they are viewed. The second are bodies of the solar system.

§ 33. The fixed stars are, therefore, beyond limits at which objects cease

to be sensibly affected by parallax. The great concave of the heavens upon which the fixed stars appear to be situated, is called the *celestial sphere.* Not only, therefore, is the longest rectilineal dimension of the earth, but also the distance between the points of its orbit about the sun most remote from each other—a distance, as we shall see in the sequel, equal to one hundred and ninety millions of miles—utterly insignificant when expressed in terms of the radius of the celestial sphere as unity. A sphere large enough to contain the entire orbit of the earth is a mere point in comparison with the vast volume embraced by the celestial sphere. *The centre of the earth may, therefore, always be regarded as the centre of the celestial sphere.*

SHAPE OF THE EARTH.

Fig. 2.



§ 34. The earth, being the station from which all the other heavenly bodies are viewed, is the first to claim attention. It has been repeatedly circumnavigated in different directions, and the portions of its surface visible from elevated positions in the midst of extended plains or at sea, always appear as circles of which the spectator seems to occupy the centre. The apparent diameters of these circles, measured by instruments, are smaller in proportion as the points of view S are more elevated. The earth is, therefore, *globular* ; for to such figures alone belong the property of always presenting to the view a circular outline.

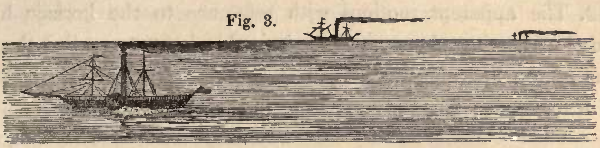
§ 35. By the figure of the earth is meant its general shape without regard to the irregularities of surface which form its hills and valleys. These are relatively insignificant and are disregarded in speaking of the earth's form. They are less in proportion to the entire earth than the protuberances and indentations on the surface of a smooth orange are to a large size specimen of that fruit. The earth is an *oblate spheroid*, and the operations and method of computations by which its precise magnitude and proportions are found, will be given presently.

The *shortest diameter* of the earth is called its *axis.*

DIURNAL MOTION.

§ 36. The boundary of the visible portion of the earth's surface, supposed perfectly smooth, is called the *sensible horizon*. The sensible horizon

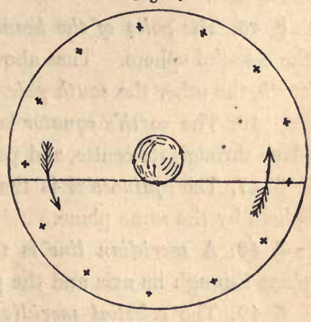
Fig. 3.



is only seen at sea, or on extended plains. At most localities on land it is broken by hills, valleys, and other objects.

§ 37. The earth conceals from us that portion of space below our sensible horizon, while all above is exposed to view. It rotates upon its axis, and the period required to perform one entire revolution is called a *day*.

Fig. 4.



§ 38. Every spectator is carried about the earth's axis in the circumference of a circle, and while the *extent* of the visible portion of space remains unchanged, different regions are continually passing through the field of view. The horizon of a spectator will be ever depressing itself below those bodies which lie in the region of space towards which he is carried by the rotation, and elevating itself above those in the opposite quarter; thus successively bringing into view the former and hiding the latter.

§ 39. The spectator being unconscious of his own motion, concludes, from first appearances, that his horizon is at rest, and attributes these changes to an actual motion in the objects themselves. Instead of his horizon approaching the bodies, he judges the bodies to approach his horizon; and when it passes and hides them, he regards them as having sunk below it or *set*, while those it has just disclosed, and from which it is receding, he considers as having come up or *risen*.

§ 40. One entire revolution about the axis being completed, the spectator returns to the place from which he commenced his observations, and he begins again to witness the same succession of phenomena and in the same order. All the heavenly bodies appear to occupy the same places in the concave sky which they did before.

§ 41. Thus the rotation of the earth about its axis produces the daily

rising and setting of the sun—the alternation of day and night; also the rising and setting of the other heavenly bodies, their progress through the vault of the heavens, and their return to the same apparent places at short and definite intervals.

§ 42. The apparent motions with reference to the horizon by which these daily recurring phenomena are brought about, are called the *diurnal motions* of the heavenly bodies. The real motion is in the horizon, the origin of reference; it is only apparent in the bodies themselves.

DEFINITIONS.

§ 43. The *axis of the celestial* sphere is the axis of the earth produced.

§ 44. The *poles of the earth* are the points in which its axis pierces its surface. The pole nearest to Greenland is called the *north*, the other the *south pole.*

§ 45. The *poles of the heavens* are the points in which its axis pierces the celestial sphere. That above the north pole of the earth is called the *north*, the other the *south pole.*

§ 46. The *earth's equator* is the intersection of the earth's surface by a plane through its centre, and perpendicular to its axis.

§ 47. The *equinoctial* is the intersection of the surface of the celestial sphere by the same plane.

§ 48. A *meridian line* is the intersection of the earth's surface by a plane through its axis and the place of a spectator.

§ 49. The *celestial meridian* is the intersection of the surface of the celestial sphere by the same plane. This is often called simply the *meridian* of the place.

§ 50. The poles of the celestial meridian are called the *East* and *West* points; that towards which the spectator is moving by his diurnal motion being the East, that from which he is receding the West.

§ 51. The *apparent zenith* and *apparent nadir* are the points in which a plumb-line produced intersects the celestial sphere: that over head being the zenith.

§ 52. The *rational horizon* is the intersection of the celestial sphere by a plane through the earth's centre and perpendicular to the line of the zenith and nadir. The plumb-line being always normal to the earth's surface, the plane of the rational horizon is parallel to the plane tangent to the earth's surface at the spectator's place, and these planes intersect the celestial sphere sensibly in the same great circle.

§ 53. The *dip of the horizon* is the angle which the elements of a

visual cone, whose vertex is in the eye of the spectator, and whose surface is tangent to that of the earth along the sensible horizon, make with the tangent plane to the earth at the spectator's place. The dip is greater in proportion as the spectator's elevation above the earth is greater. When the eye is in the earth's surface, the dip is zero, and the visual cone becomes the tangent plane. This coincidence will always be supposed to exist unless the contrary is specially noticed.

§ 54. The *latitude* of a place on the earth's surface is the arc of the celestial meridian from the equinoctial to the zenith of the place. It is always measured in degrees, minutes, seconds, and thirds. Latitude is reckoned north or south; that reckoned towards the north pole being called north latitude, that towards the south pole, south latitude. The greatest latitude a place can have is 90°, this being the latitude of the poles of the earth.

§ 55. *Parallels of latitude* are small circles on the earth's surface parallel to the equator. All places on the same parallel have the same latitude.

§ 56. The *longitude* of a place on the earth's surface is the arc of the equinoctial intercepted between the meridian of the place and that of some other place assumed as a first meridian. It is called East or West, according as it is reckoned in the direction from the first meridian towards its east or west point. For the sake of uniformity, it will, in the text, always be reckoned in the latter direction. The English estimate longitude from the meridian of Greenwich, the French from that of Paris, and other nations from other meridians. In the United States, for most geographical purposes, it is estimated from the meridian of Washington.

§ 57. A *vertical circle* is the intersection of the celestial sphere by a plane through the zenith and nadir.

The *prime vertical* is the vertical circle whose plane is perpendicular to that of the meridian.

§ 58. The *north* and *south points* are the poles of the prime vertical; that below the north pole being called the north point.

§ 59. The *Azimuth* of a body is the angle which a vertical circle through the body's centre makes with the meridian. It is measured on the horizon, and from the south towards the west, or from the north towards the west, according as the north or south pole is elevated above the horizon. It may vary from 0° to 360°.

§ 60. The *zenith distance* of an object is the angular distance from the *apparent* zenith to the centre of the object, measured on a vertical circle.

§ 61. The *altitude* of an object is the angular distance from the horizon to the object's centre, measured on a vertical circle.

The azimuth and zenith distance are a species of polar co-ordinates for the designating an object's place in the heavens. By making the azimuth vary from zero to 360°, and the zenith distance from zero to 90°, every visible point of celestial space may be defined in position.

§ 62. A *declination circle*, or *hour circle*, is the intersection of a plane through the axis of the heavens with the celestial sphere.

§ 63. The *declination* of an object is the angular distance of its centre from the equinoctial, measured on a declination circle. The declination may be north or south, and may vary from 0° to 90°.

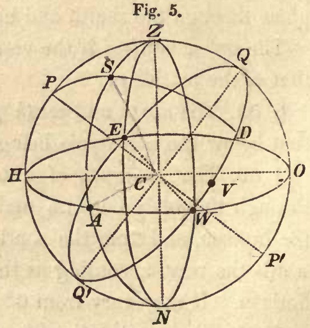
§ 64. The *polar distance* of an object is the angular distance of its centre from the celestial pole, measured on a declination circle.

§ 65. The *right ascension* of an object is the angle which a declination circle through the object's centre makes with a declination circle through a certain point on the equinoctial, called the *Vernal Equinox.* This angle is measured upon the equinoctial, and eastwardly in direction.

§ 66. The polar distance and right ascension are also a kind of polar co-ordinates for defining the places of celestial objects; for this purpose it is only necessary to cause the right ascension to vary from 0° to 360°, and the polar distance to vary from 0° to 180°, to reach every point in the celestial sphere.

§ 67. The *hour angle* of an object is the angle which its hour circle makes with the meridian of the place. It is estimated from the meridian westwardly, and may vary from 0 to 360°. The hour angle may be employed, instead of the right ascension, with the polar distance to define an object's place.

Fig. 5.



To illustrate, let the plane of the paper be that of the meridian; the circle $HZON$ its intersection with the celestial sphere; PP' the axis of the heavens; P and P' the north and south poles respectively; Z and N the zenith and nadir, respectively, and the earth a mere point at C; then will the circle $QWQ'E$, of which P and P' are the poles, be the equinoctial; $HWOE$, of which Z and N are the poles, the horizon; E and W, the poles of the meridian, will be the east and west points respectively; the arc ZQ will be the latitude, ZSA a vertical circle,

$Z S$ the zenith distance of the object S, $A S$ its altitude, and $O W A$ its azimuth; $P S$ will be its polar distance, $D S$ its declination, $Z P S$, measured by $Q D$, its hour angle, and if V be the vernal equinox, $V D$ will be its right ascension.

INSTRUMENTS.

§ 68. Most of the data with which the practical astronomer labors, come from measurements made in the circles just referred to, by means of certain astronomical instruments. These instruments are described, and their theory, adjustments, and uses explained, in Appendix II. The student should study, in connection with short daily lessons of the text, from this point, the Clock, Chronometer, Transit, Mural Circle, and Azimuth and Altitude Instrument. The others should be taken up where referred to, in the order of the text.

PROPORTIONS OF LAND AND WATER.—THE ATMOSPHERE.

§ 69. To resume the consideration of the earth. About three fourths of its surface are covered with water, and the greatest depth of the sea does not probably exceed the greatest elevation of the continents.

The earth is surrounded by a gaseous envelope, called the atmosphere, the actual thickness of which, were it reduced to a uniform density throughout, equal to that at the surface of the sea, would be about five miles. But owing to the law which regulates the pressure, density, and temperature of elastic bodies, it is much greater than this. The different strata, being relieved from the weight of those below them, become more expanded in proportion as they are higher, and the place of the superior atmospheric limit must result from an equilibrium between the weight of the terminal stratum and the elastic force of that upon which it rests. The laws just referred to indicate that this limit cannot be much higher than 80 miles.

§ 70. The atmosphere is not perfectly transparent. The sun illumines its particles; these scatter by reflection the light they receive, particularly the blue, in all directions, and produce that general illumination called daylight, and gives to the sky its bluish aspect. But for this diffusive power of the air, no object could be visible out of direct sunshine; the shadow of every passing cloud would be pitchy darkness, the stars would be visible all day, and every apartment into which the sun did not throw his direct rays would be involved in total obscurity. In ascending to

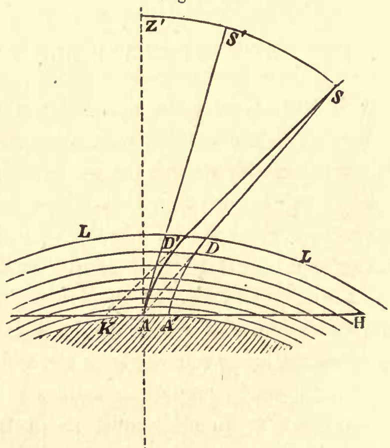
the summits of high mountains, the diffused light becomes less and less, the sky deepens in hue, and finally, at great altitudes, approaches to total blackness.

§ 71. The superior illumination of the atmosphere produced by the solar light obliterates, as it were by contrast, the light from almost all the other heavenly bodies, and few, if any, of the latter are seen when the sun is up.

REFRACTION.

§ 72. Luminous waves which enter the atmosphere obliquely are, according to the laws of optics, deviated by the latter from their course, and made to exhibit the objects from which they proceed in positions different from those they actually occupy, and thus false impressions are produced in regard to true places of the heavenly bodies.

Fig. 36.



Take, for example, a spectator on the earth at A; and let LDL represent a section of the superior limit of the atmosphere, and KAA' that of the earth's surface by a vertical plane. A star at S would, in the absence of the atmosphere, appear in the direction AS; but in reality, when the portion of the luminous wave moving on this line reaches the point D, it is turned downward, and made to come to the earth at some point A', pursuing a course such as to bring its successive positions normal to some curve, as DA', whose curvature increases towards the earth's surface, in consequence of the increasing density of the atmosphere in that direction. This part of the wave cannot therefore go to the spectator. Not so, however, with a portion of the same general wave incident at some point as D', nearer to the zenith; this, after pursuing a path $D'A$ similar to DA', will reach the spectator at A, and cause the body from which it originally proceeded to appear in the direction AS', tangent to the curve at the point A, the effect being the same as though the body had shifted its place towards the zenith by the angular distance SAS'.

§ 73. The air's refraction, therefore, diminishes apparently the zenith

distances of all bodies, and increases their altitudes. Any body actually in the horizon will appear above it, and any body apparently in the horizon must be below it.

§ 74. It is also obvious that refraction can only take place in the vertical plane through the body, since this plane is always normal to the surfaces of the atmospheric strata, and divides them symmetrically. Refraction will not, therefore, in general, affect the azimuth of a body.

§ 75. This apparent angular displacement of a body from its true place, caused by the action of the atmosphere upon its luminous waves, is called *refraction;* and various formulas have been constructed to compute its exact amount. One of the best of these is by Littrow, which has the merit of depending upon no special hypothesis in regard to the constitution of the atmosphere, being constructed upon the most general principles, and from known and well-ascertained data.

§ 76. Make,

$Z = Z'AS' =$ observed zenith distance;
$r = SAS' =$ corresponding refraction;
$h =$ height of mercurial column, which the atmosphere supports;
$t =$ temperature of the air and of the mercury;
$\alpha =$ coefficient of atmospheric expansion for each degree of Fahr.;
$\beta =$ coefficient of expansion for mercury, same thermometric scale

Then, Appendix No. III.,

$$r = 57''.82 \,.\, \frac{h}{30} \,.\, \frac{1+(50-t)\beta}{1+(t-50)\alpha} \,.\, \tan Z \,.\left(1 - 0.0012517 \sec^2 Z + 0.00000139 \frac{2+\sin^2 Z}{\cos^4 Z}\right) \quad (2)$$

or, omitting the last term in the parenthesis as being insignificant for ordinary zenith distances,

$$r = 57''.82 \,.\, \frac{h}{30} \,.\, \frac{1+(50-t)\beta}{1+(t-50)\alpha} \,.\, \tan Z \,.\, (1 - 0.0012517 \sec^2 Z) \; . \, . \quad (3)$$

When $h = 30$, and $t = 50$, equation (3) becomes

$$r_m = 57''.82 \tan Z\,(1 - 0.0012517 \sec^2 Z) = A \quad . \quad . \quad (4)$$

and the results given by this formula for different values for Z are called mean refractions; and for any other state of the thermometer and barometer.

$$r = A \,.\, \frac{h}{30} \,.\, \frac{1+(50-t)\beta}{1+(t-50)\alpha},$$

and taking logarithms,

$$\log r = \log A + \log \frac{h}{30} + \log \frac{1+(50-t)\beta}{1+(t-50)\alpha} \quad . \quad . \quad . \quad (5)$$

Causing Z to vary from 0° to 90°, h from 28 to 31 inches, and t from 80° to 20°, the logarithms above may be computed and tabulated for future use, under the heads Z, t, and b.

§ 77. Causing Z to vary from 0° to 90°, in equation (4), we may construct Table I.; causing t to vary from 80° to 20°, and h to vary from 31 to 28, in the last two terms of equation (5), we may construct Table II. Returning to equation (2), resuming the quantity omitted to obtain equation (3), computing their values for zenith distances, varying from 75° to 90°, on the supposition that $h=30$ and $t=50$, an additional table may be computed to correct the refractions in low altitudes. Tables I., II., and III. are due to Mr. Ivory.

§ 78. For zenith distances exceeding 80°, refraction becomes very uncertain; it then no longer depends solely upon the state of the atmosphere, which is indicated by the barometer and thermometer, being frequently found to vary at the same station some 3 to 4 minutes for the same indications of these instruments.

Example.—The zenith distance of an object is observed to be 71° 26′ 00″, the barometer standing at 29.76 in., and the thermometer at 43° Fahr: required the refraction.

Table I.	Mean refraction,		log. 2.23609
Table II.	Barometer 29.76		" 9.99651
Table II.	Thermometer 43°		" 0.00668
Refraction		2′ 53″.49 . .	2.23928
Observed zenith distance . . .		71° 26′ 00″.00	
Zenith dist. cleared from refraction		71° 28′ 53″.49	

The refraction must always be added to the observed zenith distance, or subtracted from the observed altitude, to clear an observation from refraction.

PARALLELISM OF THE EARTH'S AXIS, AND UNIFORMITY OF THE EARTH'S DIURNAL MOTION.

§ 79. Wherever upon the earth's surface the altitudes and instrumental azimuths of a star are taken in the various points of its diurnal course, and the instrument is turned in azimuth, so as to read the half sum of two azimuths, corresponding to any two equal altitudes, the vertical plane through the line of collimation is found to divide the path symmet-

rically; and this plane of symmetry for any one star will, at the same place of observation, also be a plane of symmetry for all the stars. In other words, the diurnal paths of the stars may be divided symmetrically by any number of planes inclined to one another through the earth's centre—a condition which can only be fulfilled for paths upon the celestial sphere, when these paths are circles, of which the poles coincide, and the planes of symmetry pass through them.

The diurnal motions of the stars are only apparent, and arise from an actual motion of the spectator about the earth's axis. This latter line preserves, therefore, its direction unchanged, and, in the motion of the earth around the sun, describes a cylindrical surface, of which the elements have their vanishing point in the poles of the celestial sphere. These poles are therefore the geometric poles of the diurnal paths of the stars, and the planes of symmetry are the meridian planes of the places of observation.

§ 80. Again, the interval of time during which a star is moving between any two given altitudes on one side of the plane of symmetry, is exactly equal to that during which it is moving between the equal altitudes on the opposite side, which can only be true, for all positions of the observer, when the star's apparent, or *the earth's real motion about its axis, is uniform.*

§ 81. The period of one revolution of the earth about its axis is called a day; the day is divided into 24 equal parts called *hours;* the hours into 60 equal parts called *minutes;* the minutes into 60 equal parts called *seconds*, and the seconds into 60 equal parts called *thirds.*

§ 82. The earth rotates therefore at the rate of $360 \div 24 = 15°$ an hour; $15'$ of space in 1 minute of time; $15''$ of space in 1 second of time, or $15'''$ of space in 1 third of time.

§ 83. Distances on the equinoctial may therefore be expressed in *time* or *space* at pleasure, the former being convertible into the latter by multiplying by 15, or the latter into the former by dividing by 15.

§ 84. To distinguish hours, minutes, and seconds in *time*, from degrees, minutes, and seconds in *arc*, the former are usually designated by the notation h, m, s, and the latter by $°$, $'$, $''$; thus an arc upon the equinoctial may be written $357° \ 39' \ 38''$, or $23^h \ 50^m \ 38^s.5$.

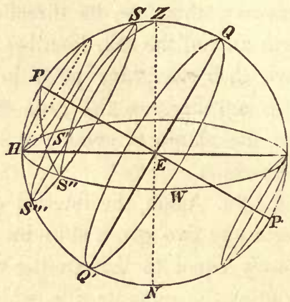
§ 85. *To find the instrumental azimuth of the meridian of a place.*—Bring the line of collimation of an altitude and azimuth instrument, properly levelled, upon a star in the east or west, clamp the vertical circle, and read the instrumental azimuth; then by an azimuthal motion bring the line of collimation upon the star when in the west or east, and again read the

azimuth: the half sum of the two will be the instrumental azimuth sought. To bring the line of collimation into the meridian, turn the instrument till it reads this half sum.

UPPER AND LOWER DIURNAL ARCS.—CIRCUMPOLAR BODIES.

§ 86. The diurnal paths of the heavenly bodies which are cut by the horizon are, in general, divided by the latter unequally. The portions of these paths above the horizon are called the *upper*, and those below the *lower diurnal arcs.*

Fig. 87.



§ 87. To find, for any spectator, the relation which these arcs bear to one another, let $P\,Q\,P'\,Q'$ be the meridian, P the elevated pole, $Q\,Q'$ the equinoctial, Z the zenith, $H\,W\,H'$ the horizon, $S'\,S\,S''\,S'''$ the diurnal path of any body, the earth being a mere point at E; then will $S'\,S\,S''$ be the upper, and $S''\,S'''\,S'$ the lower diurnal arc.

Make

$l = Q\,Z$, latitude of the spectator,

$p = P\,S''$, polar distance of the body,

$P = Z\,P\,S''$, the hour angle of the body when in the horizon,

$z = Z\,S''$, zenith distance of the body in horizon.

Then in the triangle $Z\,P\,S''$, because $P\,Z = 90 - l$,

$$\cos z = \cos p \sin l + \sin p \cos l \cos P \quad . \quad . \quad . \quad . \quad (6)$$

but $z = 90°$, whence

$$0 = \cos p \sin l + \sin p \cos l \cos P;$$

or

$$\cos P = -\frac{\tan l}{\tan p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If $l = 0$, or $p = 90°$, then will

$$\cos P = 0, \text{ and } P = 90° = 6^h;$$

that is, if the spectator be upon the equator, or the body upon the equinoctial, the semi-upper arc will be six hours, and the body will be as long above as below the horizon.

If $p < l$, then will

$$\cos P < -1;$$

which is impossible, and the place of the body can never satisfy the condition that $z = 90°$. In other words, when the polar distance is less than the latitude of the spectator's place, the body can never sink to the horizon, and will ever remain in the field of perpetual apparition. Such bodies, as well as their diurnal paths, are said to be *circumpolar*.

If $p = l$, then will

$$\cos P = -1; \; P = 180° = 12^h;$$

that is, when the polar distance of the body is equal to the latitude of the spectator's place, the body can never sink below the horizon, but will just graze it in the meridian.

If $p > l$, and $p < 90°$,

$$\cos P < 0, \; \cos P > -1; \; P > 90°, \; P > 6^h;$$

that is, all bodies between the elevated pole and the equinoctial, will be longer above than below the horizon.

If $p > l$, and $p > 90°$,

$$\cos P > 0, \; \cos P < 1, \; P < 90°, \; P < 6^h;$$

that is, if the body and the spectator be on opposite sides of the plane of the equinoctial, the semi-upper arc will be less than six hours, and the body will be a shorter time above than below the horizon.

If $p = 180° - l$, then will $\tan p = -\tan l$, and

$$\cos P = 1, \; P = 0° = 0^h;$$

that is, when the body is at a distance from the depressed pole equal to the latitude of the place, the body will never rise above the horizon, but just graze it in the meridian.

If $p > 180° - l$, then will $\tan p > -\tan l$, and

$$\cos P > 1,$$

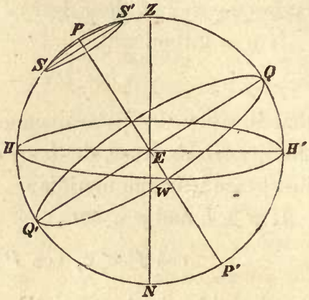
which is impossible. That is to say, if the body's distance from the depressed pole be less than the spectator's latitude, the body can never ris to the horizon, and must ever remain invisible.

§ 88. The act of a body's passing the meridian, is called its *culmination*. A body has its greatest or least altitude at the instant of its culmination. The altitude of a body when on the meridian is called its *meridian altitude*.

TERRESTRIAL LATITUDE AND LONGITUDE.

§ 89. *Latitude.*—When in Eq. (7) the angle $ZPS''=P=180°$, then will $p=l$; but in this case p is the polar distance of the point of the horizon of the same name as the elevated pole, and hence the *latitude of the spectator is always equal to the altitude of the elevated pole.*

Fig. 88.



§ 90. This suggests an easy and accurate method of getting from observation both the latitude of the spectator's place and the polar distance of a star.

Let Z be the zenith, HH' the horizon, QQ' the equinoctial, P the elevated and P' the depressed pole, and $S'S$, the diurnal path of a circumpolar star.

Make

$l = HP = ZQ$, the latitude,
$p = PS' = PS$, the polar distance of star,
$a' = HS'$, the greatest observed meridian altitude of star,
$a_{,} = HS$, the least observed meridian altitude of star,
r' and $r_{,}$, the refractions corresponding to the greatest and least meridian altitudes respectively.

Then from the figure will

$$l = \frac{a' - r' + a_{,} - r_{,}}{2} = \frac{a' + a_{,} - (r' + r_{,})}{2} \quad . \; . \; . \quad (8)$$

$$p = \frac{a' - r' - (a_{,} - r_{,})}{2} = \frac{a' - a_{,} - (r' - r_{,})}{2} \quad . \; . \; . \; . \quad (9)$$

That is to say, the latitude of the observer's place is equal to the half sum of the greatest and least meridian altitudes of a circumpolar star; and the polar distance of the star is equal to the half difference of its greatest and least meridian altitudes. Other methods for finding the latitude will be given in another place.

§ 91. *Longitude.*—The uniform motion of the earth about its axis furnishes the means of finding the longitude of the spectator's place.

Twenty-four perfect time-keepers, with dial-plates graduated to 24 hours, placed upon meridians 15° apart, and so regulated as to mark 24^h at the instant any *one fixed star* or other point of the heavens culminates, would,

§ 82, when this regulating star or point comes to any one of these meridians, simultaneously mark the hours indicated by the natural numbers from one to twenty-four, inclusive; that 15° to the east of the regulating point marking 1^h, that 30° to the east marking 2^h, and so on to that 345° to the east, or 15° to the west, marking 23^h. The timepieces to the east would be later and later, those to the west earlier and earlier. The times indicated on these several timepieces are called *the local times* of their respective meridians.

§ 92. If now, without altering its hands or rate of motion, a traveller were to transport the time-keeper of any one of these meridians to that on any other, and note the difference of time indicated by the two, this difference would be the difference of longitude of the two meridians, expressed in time; and multiplied by 15 would give the same in degrees.

§ 93. If one of these meridians be the first meridian, this difference would be the longitude of the other. But if neither be the first meridian, this difference applied to the longitude of one, supposed known, would give the longitude of the other.

§ 94. The solution of the problem of longitude consists, therefore, in *finding the difference of the local times which exist simultaneously on the first and required meridians.* The various modes of doing this will be given in another place.

FIGURE AND DIMENSIONS OF THE EARTH.

§ 95. A fluid mass rotating about an axis, and of which the particles attract one another with intensities varying inversely as the square of their distances apart, will assume the form of an oblate spheroid. Its axis of rotation will be both the shortest and a principal axis of figure. Where the angular velocity is such as to make the centrifugal force of the surface elements small in comparison with their weight, due to the attraction of the whole mass, the figure of the meridian section will, (§ 265, *Analyt. Mechanics,*) approach that of an ellipse of small eccentricity.

§ 96. The centrifugal force of a body at the equator of the earth, where it is greatest, is only about $\frac{1}{289}$th part of its weight. Observations upon the temperature of the strata composing the earth's crust, lead to the conclusion that at no great depth below its surface its materials are in a fluid state from excessive heat; and the researches of geology make it more than probable that there was a time when the earth was without solid matter. Its present irregularities of surface, forming mountains, hills, valleys, the bed of the ocean, of seas, lakes and rivers, are due to

changes subsequent to the surface induration from cooling, and as the vertical dimensions of these are insignificant in comparison with the depth to the centre of the entire mass, it is concluded that the figure of the earth is one of fluid equilibrium due to its rotary motion.

§ 97. Assuming the meridian section of the earth to be an ellipse, its eccentricity and semi-axes are found, Appendix No. IV., from the relations

$$e^2 = \tfrac{2}{3} \cdot \frac{c - c'}{c \sin^2 l_m - c' \sin^2 l'_m} \quad . \; . \; . \; . \; . \; . \; . \quad (10)$$

$$A = \frac{c}{1 - e^2}(1 - e^2 \sin^2 l_m)^{\frac{3}{2}} \quad . \; . \; . \; . \; . \; . \quad (11)$$

$$B = A \, . \sqrt{1 - e^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

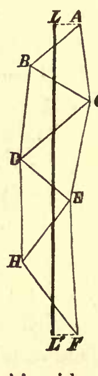
in which

e = the eccentricity of the meridian;
A = semi-transverse axis = equatorial radius of the earth;
B = semi-conjugate axis = polar radius of the earth;
c and c' = the linear dimensions of the arcs of the meridian, whose extremities differ in latitude by 1°;
l_m and l'_m = latitudes of the middle points of the arcs c and c' respectively.

The quantities l_m, l'_m, c, c' are found from observation and measurement. A method by which l and l' may be found is explained in § 90.

§ 98. To find c and c', a base line AB is carefully measured on some extended plain, and a number of stations C, D, E, F, H, &c., are selected in a northerly or southerly direction, and so that C may be seen from A and B, D from B and C, E from C and D, and so on to the end. The several stations being connected by right lines, a network of triangles is formed; every angle in each triangle is carefully measured, and the instrumental azimuth of its vertex, and that of the meridian, as viewed from the other two, accurately noted, (§ 85). The angles being cleared from *spherical excess*, the sides of the triangles are then computed, beginning of course with the triangle of which the measured base is one of the sides. The difference between the instrumental azimuths of the several vertices and those of the meridian, gives the inclination of the sides to the meridian line. The product of each side into the cosine of its inclination gives the projection of this side on the meridian, and the sum of the projections of any one of the series of sides, as AB, BC, CD, DE, EH, and HF, connecting the most north-

Fig. 39.



erly and southerly points, will give the linear meridional distance $L\,L'$, between the parallels of latitude through the same points.

Make

a = the sum of these projections, expressed in miles;

l_n = the latitude of A, supposed the most northerly;

l_s = the latitude of F, supposed the most southerly;

then,

$$l_n - l_s : 1^\circ :: a : c,$$

whence

$$c = \frac{a}{l_n - l_s},$$

and

$$l_m = \frac{l_n + l_s}{2}.$$

The same operations being repeated in a different locality considerably further north or south, the values of c' and l'_m are found, and hence from equations (10), (11), and (12), the dimensions of the earth.

From the arcs known as the Peruvian, Indian, French, English, Hanoverian, Danish, Prussian, Russian, and Swedish, names derived from the countries in which the arcs were mostly measured, Bessel found,

$$\left.\begin{array}{rl} e^2 = & 0.0068468, \\ 2\,A = & 7925{,}604 \text{ miles}, \\ 2\,B = & 7899{,}114 \text{ miles}, \\ \text{Polar compression} = & 26{,}490 \text{ miles}. \end{array}\right\} \quad \ldots\ldots\ldots \quad (13)$$

§ 99. By the *ellipticity of the earth* is meant the difference between its equatorial and polar radii, expressed in terms of the equatorial radius as unity. Denoting the ellipticity by E, we have

$$E = \frac{A - B}{A} = \frac{1}{300}, \text{ nearly} \quad \ldots\ldots \quad (14)$$

§ 100. The length of a degree of latitude, denoted by β, in any latitude l, is, Appendix No. IV, equation (l), given by

$$\beta = \frac{2\,\pi}{360^\circ} \cdot A \cdot \frac{1 - e^2}{(1 - e^2 \sin^2 l)^{\frac{3}{2}}} \quad \ldots\ldots \quad (15)$$

The length of a degree, measured perpendicularly to the meridian, denoted by β_1, is, Appendix No. IV., equation (n), given by

$$\beta_1 = \frac{2\,\pi}{360^\circ} \cdot A \cdot \sqrt{\frac{1 - e^2 \sin^2 l}{1 - e^2(2 - e^2)\sin^2 l}} \quad \ldots\ldots \quad (16)$$

and the length of a degree of longitude, denoted by α, measured on a parallel of latitude in the latitude l, is, App. No. IV., equation (o), given by

$$a = \frac{2\pi}{360} \cdot A \cdot \frac{\cos l}{\sqrt{1 - e^2 \sin^2 l}} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

§ 101. The close agreement between the results of these formulas and those of actual measurement, at various and numerous places on the earth, justifies in the fullest manner the assumption in regard to its ellipsoidal figure.

The equatorial circumference of the earth is 24,899, say, for convenience of memory, 25,000 miles. The lengths of the degrees of latitude increase from the equator to the poles. In the latitude of 50° the length is about 70 statute miles, and contains nearly as many thousand feet as the year contains days (365), and each second is equivalent to about 100 feet.

GEOCENTRIC PARALLAX.

§ 102. The bodies of the solar system being comparatively near to the earth, a change in a spectator's place on the earth's surface gives to them a sensible parallactic motion on the surface of the celestial sphere, and two observers at remote stations would not assign to these bodies the same places at the same time without first clearing their observed co-ordinates of this source of discrepancy. The mode of correction is to refer all observations to one common station, and this station is assumed, for convenience, to be at the centre of the earth.

§ 103. The place in which a body would appear, if viewed from the centre of the earth, is called its *Geocentric Place.*

§ 104. The apparent change of a body's place that would arise from a change of the spectator's station from the surface to the centre of the earth is called *Geocentric Parallax.*

§ 105. The transfer of station from the surface to the centre of the earth is sensibly in a vertical circle, and the geocentric parallax is therefore in the same plane.

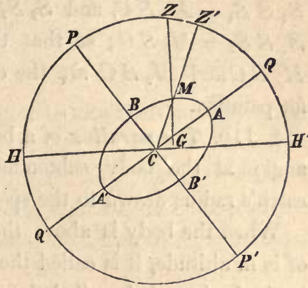
§ 106. The co-ordinates of a body's place, as determined by observation, corrected for geocentric parallax, are the geocentric co-ordinates of the body.

§ 107. The point in which the radius of the earth produced through the spectator's place pierces the celestial sphere, is called the *central zenith.* The arc of the celestial meridian from the central zenith to the equinoc

tial, is called the *central latitude.* The difference between the latitude and the central latitude, is called the *reduction of latitude.*

Thus $B A B' A'$, being a meridian section of the terrestrial spheroid, and $Z Q$ an arc of the celestial sphere in the same plane, M the spectator's place, Q the highest point of the equinoctial, $M G$ the direction of the plumb-line, $C M$ the radius of the earth; then will Z' be the central zenith, $Z' Q$ the central latitude, and $Z M Z' = Z Q - Z' Q$, the reduction of latitude.

Fig. 40.



§ 108. Denote in future the central latitude by l, the polar radius by γ, and the latitude by l', then, Appendix No. IV., equation (q),

$$\tan l = \gamma^2 \tan l' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

that is, the tangent of the central latitude is equal to the tangent of the latitude into the square of the polar radius.

Denote the radius of the earth drawn to the spectator's place by ρ, then, Appendix No. IV., equation (r),

$$\rho = \frac{1}{\sqrt{1 + \frac{1-\gamma^2}{\gamma^2} \sin^2 l}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Thus, the latitude being found from observation (§ 90), the central latitude becomes known from equation (18), and hence the radius of the earth drawn to the spectator's place, equation (19).

§ 109. Let $AB'A'B$ be a meridian section of the earth's surface, $A A'$ the equatorial diameter, M_1 and M_2 the places of two observers viewing the same body S. The observer at M_1 would see the body projected upon the celestial sphere at S_1, that at M_2 would see it projected at S_2, and to an observer at the centre it would appear at S_3. The points Z_1 and Z_2 are the central zeniths of the two observers; $Z_1 S_1$ and $Z_2 S_2$ are the central zenith distances of S, as viewed from M_1 and M_2 respectively. The first diminished by $S_3 S_1$ and the second by $S_3 S_2$, will give

Fig. 41.

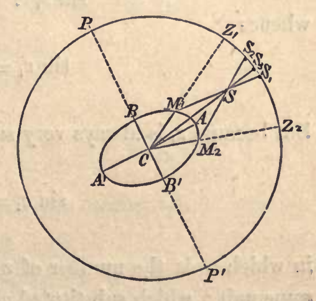
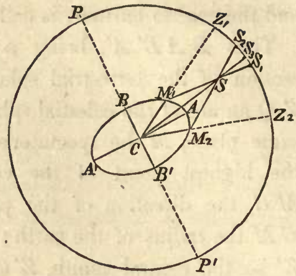


Fig. 41 bis.



$Z_1 S_3$ and $Z_2 S_3$ the central zenith distances as they would appear from the centre. But $S_3 S_1$ measures the angle $S_3 S S_1 = M_1 S C$, and $S_2 S_3$ the angle $S_2 S S_3 = M_2 S C$; so that the angles $M_1 S C$ and $M_2 S C$ are the corrections for parallax.

§ 110. The *parallax* of a body is the angle at the body subtended by the earth's radius drawn to the spectator.

When the body is above the horizon, or is in altitude, it is called the *parallax in altitude.* When the body is in the horizon, it is called the *horizontal parallax.*

Make

$z_1 = M_1 S C =$ parallax in altitude at M_1;
$z_2 = M_2 S C =$ parallax in altitude at M_2;
$P_1 =$ horizontal parallax at M_1;
$P_2 =$ horizontal parallax at M_2;
$r = C S$ = distance of the body from the earth's centre;
$\rho_1 = M_1 C$ = radius of the earth for M_1;
$\rho_2 = M_2 C$ = radius of the earth for M_2;
$Z_1 = Z_1 S_1$ = central zenith distance at M_1;
$Z_2 = Z_2 S_2$ = central zenith distance at M_2:
$l_1 = Z_1 C A =$ central latitude of M_1;
$l_2 = Z_2 C A =$ central latitude of M_2.

Then, in the triangle $M_1 S C$,

$$\sin z_1 : \sin Z_1 :: \rho_1 : r;$$

whence

$$\sin z_1 = \frac{\rho_1}{r} . \sin Z_1;$$

But because z_1 is always very small, we may write

$$\sin z_1 = \frac{z_1}{\omega};$$

in which ω is the number of seconds in radius, and z_1 is expressed in the same unit; which substituted above gives

$$z_1 = \omega . \frac{\rho_1}{r} . \sin Z_1;$$

when Z_1 becomes $90°$, the body is in the horizon, and z_1 becomes P_1, and we have

$$P_1 = \omega \cdot \frac{\rho_1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and this above gives

$$z_1 = P_1 \cdot \sin Z_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Whence the parallax in altitude is equal to the horizontal parallax into the sine of the central zenith distance.

§ 111. If the observer be upon the equator, then will ρ_1 become unity, P_1 becomes the horizontal parallax on the equator, called the *equatorial horizontal parallax;* designating this latter by P, we have, equation (20),

$$P = \frac{\omega}{r};$$

and this in equation (20) gives

$$P_1 = P \cdot \rho_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

that is to say, the horizontal parallax of a body at any place, is equal to the product of the equatorial horizontal parallax of the body by the radius of the earth at the place.

The value of P_1 in equation (21) gives

$$z_1 = P \cdot \rho_1 \cdot \sin Z_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

§ 112. To find the equatorial horizontal parallax of any body, we have in the triangles $M_1\ S\ C$ and $M_2\ S\ C$

$$z_1 = P \cdot \rho_1 \cdot \sin Z_1$$
$$z_2 = P \cdot \rho_2 \cdot \sin Z_2$$

adding

$$z_1 + z_2 = P \cdot (\rho_1 \cdot \sin Z_1 + \rho_2 \cdot \sin Z_2),$$

but

$$z_1 = Z_1 - Z_1 C S,$$
$$z_2 = Z_2 - Z_2 C S;$$

by addition

$$z_1 + z_2 = Z_1 + Z_2 - (Z_1\, C S + Z_2\, C S) = Z_1 + Z_2 - (l_1 + l_2),$$

which substituting above, and dividing by the coefficient of P, gives

$$P = \frac{Z_1 + Z_2 - (l_1 + l_2)}{\rho_1 \sin Z_1 + \rho_2 \sin Z_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

§ 113. If the body be so remote that the difference between the radii of the earth, as viewed from it, be insignificant, which is the case with all

bodies except the moon, ρ_1 and ρ_2 may be regarded as equal to one another, and each equal to unity, and we shall have, equation (24),

$$P = \frac{Z_1 + Z_2 - (l_1 + l_2)}{\sin Z_1 + \sin Z_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

in which l_1 and l_2 are the central latitudes of the places M_1 and M_2.

§ 114. In all this the observers have been supposed to be on the same meridian; but this is not necessary, nor would it, in general, be the case in practice. If on different meridians, make

δ = change of meridian zenith distance of the body in the interval between two consecutive culminations;

λ = difference of longitude of the two observers, expressed in time;

δ' = change in meridian zenith distance while passing from the first to the second meridian;

then

$$24^h : \delta :: \lambda : \delta';$$

whence

$$\delta' = \frac{\lambda . \delta}{24^h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

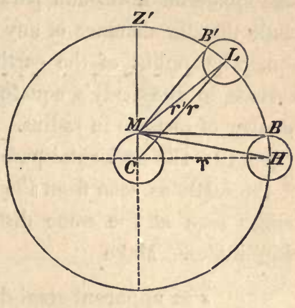
If the meridian zenith distance be increasing at the easterly station, δ' is to be added to, if decreasing, subtracted from, the meridian zenith distance at that station. This corrected meridian zenith distance will be that which the body would have to an observer on the meridian of the westerly station, and on the same parallel of latitude with the observer on the easterly meridian, the reduction being in effect to bring the observers to the same meridian.

§ 115. To recapitulate: the latitudes of two stations are first found from observation; the central latitudes are found from equation (18); the radii of the earth at the two stations, from equation (19); the equatorial horizontal parallax, from equation (24); the horizontal parallax at any place, from equation (22); and the parallax in altitude, from equation (23).

AUGMENTED AND HORIZONTAL DIAMETERS.

§ 116. By the rotation of the earth upon its axis the spectator is continually changing his distance from the heavenly bodies. A change of distance gives rise to a change in the apparent dimensions of an object. A body seen in the horizon of a spectator would appear to him sensibly of the same size as if seen from the centre of the earth, the distances HC and HM, for the nearest of the heavenly bodies, being sensibly the same.

Fig. 42.



The *apparent semi-diameter* of a body is the angle at the observer subtended by the body's real semi-diameter, the latter being perpendicular to a visual ray drawn to one of its extemities.

Make

$s = HMB =$ apparent semi-diameter of a body when in the horizon;

$s' = LMB' =$ apparent semi-diameter of the body when in altitude;

$d = HB = LB' =$ real semi-diameter of the body in linear units;

$r =$ body's distance from the observer when in the horizon $=$ distance from earth's centre;

$r' =$ body's distance from the spectator when in altitude;

$Z = Z'ML =$ the body's central zenith distance;

$z = MLC =$ the body's parallax in altitude.

Then

$$r \cdot \sin s = d = r' \sin s';$$

whence

$$\frac{\sin s'}{\sin s} = \frac{r}{r'} = \frac{\sin Z}{\sin (Z-z)} = \frac{1}{\cos z - \dfrac{\cos Z}{\sin Z} \cdot \sin z}$$

replacing $\sin z$ by its value $\frac{z}{\omega}$, and z by its value in Eq. (23), also writing $\frac{s}{\omega}$ for $\sin s$, and $\frac{s'}{\omega}$ for $\sin s'$, we find

$$\frac{s'}{s} = \frac{1}{\cos z - \rho_1 \cdot \dfrac{P}{\omega} \cdot \cos Z} \quad \cdot \cdot \cdot \cdot \cdot \cdot \quad (27)$$

in which s' and s are expressed in seconds of arc.

§ 117. The apparent diameter $2s$ of a body in the horizon, is called the *horizontal diameter*; its apparent diameter $2s'$ in altitude, is called the *augmented diameter*.

DISTANCES AND DIMENSIONS OF THE HEAVENLY BODIES.

§ 118. Having found the horizontal parallax of a body, it is easy to find its distance from the earth's centre. From equation (20) we have

$$r = \rho_1 \cdot \frac{\omega}{P_1} = \rho \cdot \frac{\omega}{P} \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (28)$$

in which ρ and P are respectively the equatorial radius of the earth and

the equatorial horizontal parallax of the body; and from which we conclude that the distance of any body from the earth's centre, is equal to the equatorial radius of the earth repeated as many times as the number of seconds in the body's equatorial horizontal parallax is contained in the number of seconds in radius.

§ 119. The horizontal parallax of a body is the apparent semi-diameter of the earth as seen from the body. The apparent semi-diameter of two bodies seen at the same distance are directly proportional to their real magnitudes. Make

$s =$ apparent semi-diameter of the body;
$d =$ the real semi-diameter of the body in linear units, as miles;
$P =$ the equatorial horizontal parallax of the body;
$\rho =$ the equatorial radius of the earth;

then will

$$P : s :: \rho : d;$$

whence

$$d = \rho \cdot \frac{s}{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

that is, the real semi-diameter of any heavenly body is equal to the equatorial radius of the earth repeated as many times as the body's equatorial horizontal parallax is contained in its apparent semi-diameter. The apparent diameter of a body is measured by means of the micrometer.

ECLIPTIC.

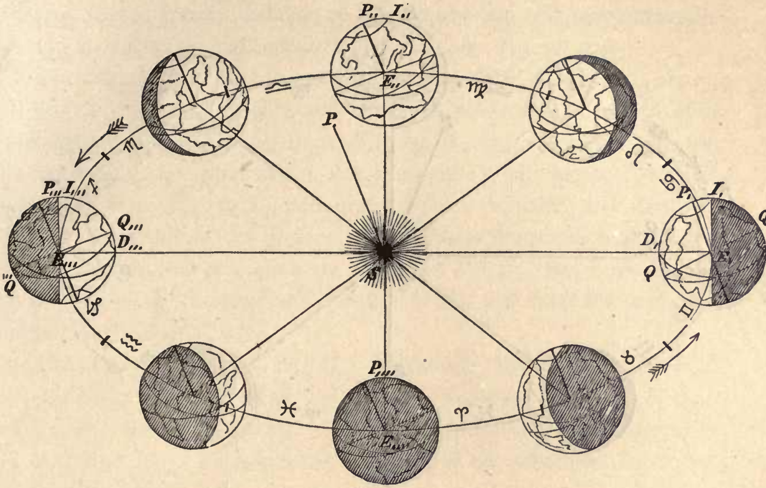
§ 120. The orbit of the earth about the sun is sensibly a plane curve. The intersection of the plane of the earth's orbit with the celestial sphere is called the *ecliptic*. The ecliptic is a great circle of the celestial sphere because its plane passes through the earth's centre.

§ 121. The orbital motion of the earth about the sun gives rise to a parallactic motion of the sun about the earth, and the effect to a spectator on the earth is the same as though the latter were stationary and the sun in motion about the earth. The sun appears to move along the ecliptic in the same direction that the earth's projection upon the celestial sphere, as seen from the sun, actually moves in that great circle.

§ 122. The earth's axis being oblique to the plane of the ecliptic, forms an angle with the radius vector of the earth. The axis of the earth retaining its direction sensibly unchanged, this angle is variable.

§ 123. Let S be the sun, $E_{,} E_{,,} E_{,,,} E_{,,,,}$ the earth's orbit, $P S$ a line through the sun's centre and parallel to the direction of the earth's axis.

Fig. 43.



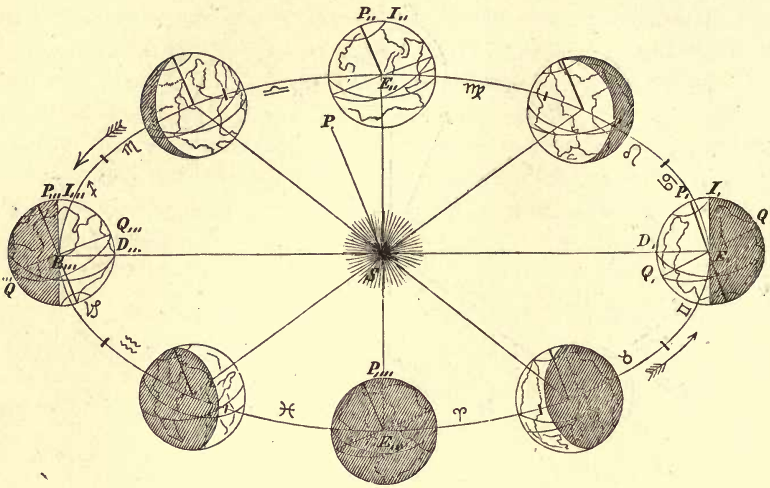
$E_{,}E_{,,,}$ the projection of this line on the plane of the ecliptic. Draw $E_{,,}SE_{,,,,}$ perpendicular to $E_{,}E_{,,,}$, and $E_{,}P_{,}$, $E_{,,}P_{,,}$, $E_{,,,}P_{,,,}$, and $E_{,,,,}P_{,,,,}$ parallel to SP. The angle $P_{,}E_{,}S$ will be the polar distance of the sun when the earth is at $E_{,}$, $P_{,,}E_{,,}S$ when at $E_{,,}$, $P_{,,,}E_{,,,}S$ when at $E_{,,,}$, and $P_{,,,,}E_{,,,,}S$ when at $E_{,,,,}$. It will be least at $E_{,}$, greatest at $E_{,,,}$, and 90° at $E_{,,}$ and $E_{,,,,}$. The polar distance at $E_{,,,}$ is the supplement of that at $E_{,}$, and estimated from the nearest or opposite poles, the polar distances at $E_{,}$ and $E_{,,,}$ are equal.

§ 124. The declination being the complement of the polar distance, the sun's declination will sometimes be north, sometimes south. Its north declination will be greatest when its north polar distance is least; its south declination greatest when its south polar distance is least.

§ 125. Thus, by the orbital motion of the earth, the terrestrial equator is carried from one side of the sun to the opposite, and the sun itself made apparently to pass alternately from north to south and from south to north of the equinoctial.

§ 126. The radius vector would, by the orbital motion alone, trace upon the surface of the earth an ellipse of which the plane would coincide with that of the ecliptic; by the diurnal motion alone, it would trace a parallel of latitude; and by both motions combined, it actually describes a kind of spiral curve extending on either side of the equator and intersecting all the parallels between those whose latitudes are equal to the sun's greatest north and south declinations. To spectators situated somewhere on these

Fig. 43 bis.



parallels, the sun will be vertical, or in the zenith, twice in the course of one revolution of the earth about the sun.

§ 127. Two small circles of the celestial sphere parallel to the equinoctial and at a distance therefrom equal to the sun's greatest north and south declination are called tropics; that on the north is called the *tropic of Cancer*, and that on the south the *tropic of Capricorn.*

§ 128. A plane through the earth's centre, and perpendicular to the radius vector, divides the earth's surface into two equal parts. That on the side towards the sun is illuminated, while that on the opposite side is in the dark. To an observer on the former it is *day;* to one on the latter it is *night.* The curve which separates the enlightened portion from the dark is called the *circle of illumination.*

§ 129. When the earth is in either of the positions $E_{,,}$ or $E_{,,,,}$, its axis is in the plane of the circle of illumination; this latter divides all parallels equally, and the lengths of the days are equal to those of the nights all over the earth's surface.

When the earth is in either of the positions $E_{,}$ or $E_{,,,}$, its axis makes the greatest angle possible with the plane of the circle of illumination; the latter divides the parallels most unequally, and the length of the days will differ from those of the nights the most possible.

§ 130. To all places north of the parallel whose latitude is equal to the north polar distance of the sun, the sun will, Eq. (7), be circumpolar, while to all places having an equal south latitude the sun will not, Eq. (7),

rise. In like manner, to all places south of the parallel whose latitude is equal to the south polar distance of the sun, the sun will be circumpolar; while to all places of equal north latitude, the sun will not rise.

§ 131. During the time the earth is moving from $E_{,,}$ to $E_{,,,,}$, the sun will shine upon the south pole, and the north pole will be deprived of his direct light. While moving from $E_{,,,,}$ to $E_{,,}$ the reverse will be true. The zones of polar illumination and obscuration will increase from $E_{,,}$ to $E_{,,,}$ and from $E_{,,,,}$ to $E_{,}$, and diminish from $E_{,}$ to $E_{,,}$ and from $E_{,,,}$ to $E_{,,,,}$. The radii of the greatest zones of polar illumination and obscuration for one diurnal revolution are $P_{,}I_{,}$ and $P_{,,,}I_{,,,}$, which are equal to one another, being the greatest departure of the pole from the circle of illumination on opposite sides.

§ 132. Draw $E_{,}Q_{,}$ and $E_{,,,}Q_{,,,}$ respectively perpendicular to $P_{,}E_{,}$ and $P_{,,,}E_{,,,}$; then will $Q'Q_{,}$ and $Q'''Q_{,,,}$ represent the equator, $Q_{,}D_{,}$ and $Q_{,,,}D_{,,,}$ the greatest north and south declinations of sun respectively, and $P_{,}I_{,}$ and $P_{,,,}I_{,,,}$ the greatest departure of the pole from the circle of illumination. Now

$$I_{,}D_{,} = P_{,}Q_{,} = 90° = I_{,,,}D_{,,,} = P_{,,,}Q_{,,,},$$
$$P_{,}D_{,} = P_{,}D_{,}, \qquad P_{,,,}D_{,,,} = P_{,,,}D_{,,,},$$

and by subtraction

$$P_{,}I_{,} = D_{,}Q_{,}, \qquad P_{,,,}I_{,,,} = Q_{,,,}D_{,,,};$$

that is, the radius of the zone of greatest polar illumination, or obscuration, is equal to the greatest declination of the sun.

§ 133. Two small circles parallel to the equinoctial, and at a distance from the poles equal to the greatest declination of the sun, are called *polar circles;* that about the north pole is called the *arctic*, and that about the south the *antarctic* circle. The polar circles are the boundaries of the greatest zones of polar diurnal illumination and obscuration.

§ 134. When the intervals of time between three consecutive passages of a circumpolar star over the line of collimation of a transit or mural circle are equal, these instruments are adjusted to the meridian.

§ 135. The diurnal motion brings the meridian of a place, in the course of one revolution of the earth on its axis, into coincidence with the declination circle of every body in the heavens. The difference of times between the meridian's passing the centres of any two bodies, is the difference of right ascension of these bodies.

§ 136. To find the time of the meridian's passing the centre of any body, find by the transit instrument and timepiece the time of the meridian's passing the body's east and west limb, and take half the sum.

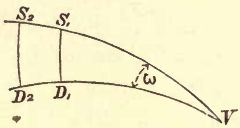
§ 137. To find the polar distance of a body's centre, take the reading of the mural circle when its line of collimation is upon the upper or lower limb; subtract from this the polar reading and correct the difference for refraction, parallax in altitude, and semi-diameter. The declination is obtained by subtracting the polar distance from 90°.

§ 138. The points in which the equinoctial intersects the ecliptic are called the *equinoxes;* that by which the sun passes from the south to the north of the equinoctial is called the *vernal equinox;* the other, or that by which the sun passes from the north to the south of the equinoctial, is called the *autumnal equinox.*

§ 139. The angle which the equinoctial makes with the ecliptic is called the *obliquity of the ecliptic.*

§ 140. To find the place of the vernal equinox and the obliquity of the ecliptic, let VD_2 be an arc of the equinoctial, VS_2 of the ecliptic, V the vernal equinox, S_1 and S_2 two places of the sun when on the meridian at different times, S_1D_1, S_2D_2 arcs of declination circles; and make

Fig. 44.



$\delta_1 = D_1 S_1$, the sun's declination at any meridian passage;
$\delta_2 = D_2 S_2$, the same at some subsequent passage;
$2a = VD_2 - VD_1$, the corresponding difference of right ascension;
$x = VD_1$, the right ascension of the sun at the time of first meridian passage;
$\omega = S_1 VD_1$, the obliquity of the ecliptic.

Then in the triangles $S_1 VD_1$ and $S_2 VD_2$, right-angled at D_1 and D_2,

$$\sin x = \tan \delta_1 \cdot \cot \omega,$$
$$\sin (x + 2a) = \tan \delta_2 \cdot \cot \omega;$$

and by division

$$\frac{\sin (x + 2a)}{\sin x} = \frac{\tan \delta_2}{\tan \delta_1} \quad \ldots\ldots\ldots \quad (30)$$

adding unity and clearing the fraction, then subtracting unity and clearing, and dividing one result by the other, we find

$$\frac{\sin (x + 2a) + \sin x}{\sin (x + 2a) - \sin x} = \frac{\tan \delta_2 + \tan \delta_1}{\tan \delta_2 - \tan \delta_1};$$

$$\frac{\tan (x + a)}{\tan a} = \frac{\sin (\delta_2 + \delta_1)}{\sin (\delta_2 - \delta_1)}.$$

Whence

$$\tan (x + a) = \frac{\sin (\delta_2 + \delta_1)}{\sin (\delta_2 - \delta_1)} . \tan a \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

Also

$$\cot \omega = \sin x . \cot \delta_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

The value of the obliquity is thus found to be nearly 23° 27′ 54″, which is therefore the greatest north and south declination of the sun. The tropics are, therefore, 23° 27′ 54″ from the equinoctial, and the polar circles are at the same distance from the poles.

§ 141. The interval of time between the sun and a star crossing the meridian, applied to the right ascension of the sun, gives the right ascension of the star. The declination of a star is found like that of the sun, except that there is no correction for parallax and semi-diameter, the only correction being for refraction.

§ 142. The right ascension and declination of one star being known, the differences of observed right ascensions and declinations, the latter being corrected for differences of refractions, give, when applied to the right ascension and declination of the known star, the right ascension and declination of other stars. Thus a list of the stars, together with their right ascensions and declinations, and arranged in the order of their right ascensions, furnishes the ground-work of what is called a catalogue of stars, of which a fuller account will be given presently.

§ 143. A belt of the heavens extending on either side of the ecliptic, far enough to embrace the paths of the planets, is called the *zodiac.*

§ 144. The ecliptic is divided into twelve equal parts, called *signs.* They commence at the vernal equinox, and are named in order, proceeding towards the east, *Aries* (♈), *Taurus* (♉), *Gemini* (♊), *Cancer* (♋), *Leo* (♌), *Virgo* (♍), *Libra* (♎), *Scorpio* (♏), *Sagittarius* (♐), *Capricornus* (♑), *Aquarius* (♒), and *Pisces* (♓). Motion in the order of the signs is said to be *direct;* the converse, *retrograde.*

§ 145. The points of the ecliptic in which the sun reaches his greatest north and south declination are called the *solstitial points:* that on the north is called the *summer solstice*, and that on the south the *winter solstice.* The sun when in these points appears to be stationary as regards his apparent motion in declination. The *solstitial colure* is the declination circle through the solstitial points. The *equinoctial colure* is the declination circle through the equinoctial points. The solstitial colure separates Gemini from Cancer, and Sagittarius from Capricornus; the equinoctial colure separates Aries from Pisces, and Virgo from Libra.

§ 146. A great circle of the celestial sphere passing through the poles of the ecliptic is called a *circle of latitude.*

§ 147. The *latitude* of a body is the distance of the body's centre from the ecliptic, measured on a circle of latitude.

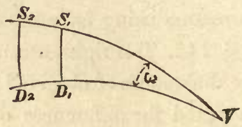
§ 148. The *longitude* of a body is the distance from the vernal equinox to the circle of latitude through the body's centre, measured on the ecliptic in the order of the signs.

The longitude and latitude are co-ordinates that refer a body's place to the circle of latitude through the vernal equinox and to the ecliptic; the longitude and ecliptic polar distance are polar co-ordinates that refer a body's place to the same circle of latitude and to the pole of the ecliptic.

§ 149. The longitude of the sun, as seen from the earth, is readily obtained from the obliquity of the ecliptic and either the right ascension or declination.

For this purpose make

Fig. 44 bis.



$\alpha = VS_1$, the longitude of the sun;
$\delta = S_1 D_1$, his declination;
$a = VD_1$, his right ascension;
$\omega = S_1 VD_1$, the obliquity of the ecliptic.

Then will

$$\tan \alpha = \frac{\tan a}{\cos \omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$\sin \alpha = \frac{\sin \delta}{\sin \omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

§ 150. The place of the sun as seen from the earth, and that of the earth as seen from the sun, are at the opposite extremities of the same diameter of the ecliptic; and the longitude of the sun, increased by 180°, will be the longitude of the earth as viewed from the sun, the centre of the earth's orbital motion.

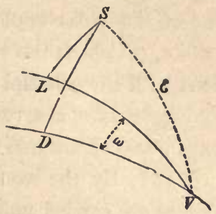
§ 151. The sun appears in the vernal equinox on the 20th March, in the autumnal equinox on the 22d September, the summer solstice on the 21st June, and in the winter solstice on the 21st December.

The poles of the ecliptic are at a distance from the nearest poles of the equinoctial, equal to the obliquity of the ecliptic.

§ 152. The right ascension is obtained from observation by means of the clock and transit instrument, the declination by means of the mural circle. From these and the obliquity of the ecliptic, the longitude and latitude are obtained from computation. Thus, let S be the body's place, V the vernal equinox, VD the body's right ascension, DS its declina-

tion, VL its longitude, SL its latitude, and the angle LVD the obliquity of the ecliptic.

Fig. 45.



Make

$\alpha = VD =$ right ascension;

$\delta = DS =$ declination;

$l = VL =$ longitude;

$\lambda = LS =$ latitude;

$\omega = LVD =$ obliquity of the ecliptic;

$\varepsilon = SV =$ arc of great circle through the body and vernal equinox;

$\varphi = SVD =$ inclination of SV to the equinoctial

Then in the triangle SVD, right-angled at D,

$$\tan \varphi = \frac{\tan \delta}{\sin \alpha} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$$\cos \varepsilon = \cos \alpha \, . \cos \delta \quad . \quad . \quad . \quad . \quad . \quad (36)$$

and in the triangle VLS, right-angled at L,

$$\tan l = \tan \varepsilon \, . \cos (\varphi - \omega) \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$\sin \lambda = \sin \varepsilon \, . \sin (\varphi - \omega) \quad . \quad . \quad . \quad . \quad . \quad (38)$$

§ 153. If the longitude, latitude, and obliquity be given, then in the triangle VLS,

$$\tan (\varphi - \omega) = \frac{\tan \lambda}{\sin l} \quad . \quad . \quad . \quad . \quad . \quad (39)$$

$$\cos \varepsilon = \cos l \, . \cos \lambda \quad . \quad . \quad . \quad . \quad . \quad (40)$$

and in the triangle VDS,

$$\tan \alpha = \tan \varepsilon \, . \cos \varphi \quad . \quad . \quad . \quad . \quad . \quad (41)$$

$$\sin \delta = \sin \varepsilon \, . \sin \varphi \quad . \quad . \quad . \quad . \quad . \quad (42)$$

PRECESSION AND NUTATION.

§ 154. The longitudes and latitudes of the stars, being thus determined at different epochs, show a slow increase in all the longitudes, while the latitudes remain sensibly the same.

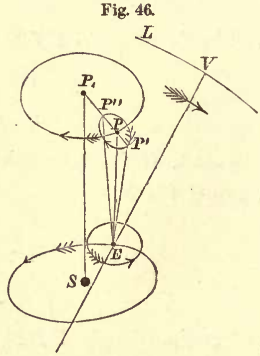
§ 155. This is owing to a slow gyratory motion of the line of the terrestrial poles in a retrograde direction, caused by the rotary motion of the earth and the combined action of the sun, moon, and planets upon the ring of equatorial matter that projects beyond the sphere of which the polar axis is the diameter. It is the resultant of two component motions.

§ 156. By the first of these components alone, called *nutation*, the line of the poles would describe once in every 19 years an acute conical surface, of which the vertex is at the centre of the earth, and the intersections with the celestial sphere are two equal ellipses, whose transverse and conjugate axes are respectively $18''.5$ and $13''.74$, the former being always directed towards the poles of the ecliptic

§ 157. By the second, called the *mean precession*, the centres of these ellipses are carried uniformly around the poles of the ecliptic from east to west in equal circles, of which the radii are about $23° 28'$, and at a rate of $50''.2$ in the interval of time between two consecutive returns of the sun to the mean vernal equinox. This interval is called a *tropical* year. The mean equinoxes perform, therefore, one entire revolution in $360° \div 50''.2 = 25817$ tropical years.

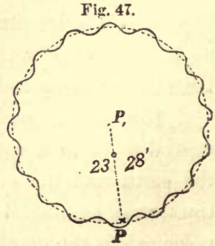
In the figure, S is the sun, E the earth, $P_{\prime}$ the north pole of the ecliptic, P' the true and P the mean north pole of the equinoctial. The curve about S represents the earth's orbit, that about E, and of which the plane is perpendicular to EP', shows the direction of the earth's axial motion; EV is the intersection of the plane of this circle with that of the ecliptic, and V is the vernal equinox. The circle about $P_{\prime}$ has a radius of about $23° 28'$, the curve about P is the elliptical path described by the true pole P' about the mean P, and of which the longer axis $P'P''$ passes through $P_{\prime}$. The arc VL of the ecliptic, the circle about $P_{\prime}$, and ellipses about P are all on the surface of the celestial sphere, while S, E, and the curves about them, are at its centre.

Fig. 46.



§ 158. By the component motions of mean precession and of nutation combined, the true equinoctial pole is carried with a variable motion along a gently waving curve whose undulations extend to equal distances on either side of the circumference of mean precession, and which it intersects at points separated by angular distances, as seen from the centre of the celestial sphere, equal to $19 \times 50''.2 \times \sin 23° 28' \div 2 \pm 13'',74 = 3' 10'' \pm 13'',74$.

Fig. 47.



§ 159. The motions due to the action of the sun and moon are opposed to those arising from the action of the planets, and when estimated along the

ecliptic are called *luni-solar precession in longitude.* The combined effect arising from the simultaneous action of all the bodies, estimated in the same direction, is called the *general precession in longitude.*

§ 160. The equinoxes always conforming to the places of the equinoctial poles, have a slow, irregular, but continuous retrograde motion.

The place of the vernal equinox without nutation is called the *mean equinox;* with nutation, the true or *apparent equinox.*

The inclination of the equinoctial to the ecliptic without nutation is called the *mean obliquity;* with nutation, the true or *apparent obliquity.* The difference between the mean and apparent obliquity is called the *nutation of obliquity.*

§ 161. The apparent equinox wanders in either direction from the mean to a distance equal to $13'',74 \div 2 \cdot \sin 23° \, 28', = 17'', 25$, which it reaches when the mean and apparent obliquity are equal; and the apparent obliquity varies on either side of the mean from zero to half of $18''.5$ or $9''.25$; the latter being reached when the apparent equinox coincides with the mean.

§ 162. The motion of the mean equinox along the ecliptic is determined by that of the centre of the little ellipse above referred to, and is therefore at the rate of $50''.2$ a year, being the quotient which results from dividing $360°$ by the period required for the true pole to perform one entire circuit around the pole of the ecliptic.

§ 163. The distance from the mean to the apparent equinox is called the *equation of the equinoxes in longitude.*

§ 164. The intersection of a declination circle through the mean equinox with the equinoctial, is called the *reduced place of the mean equinox.*

§ 165. The distance from the reduced place of the mean equinox to the apparent equinox, is called the equation of the *equinoxes in right ascension.*

§ 166. The changes which take place in these equations, as also in the apparent obliquity of the ecliptic, are called *periodical variations*, from the circumstance of their running through all their possible values in a comparatively short period.

Formulas for computing the equations of the equinoxes in longitude and right ascension will be given in another place.

§ 167. Besides the motion of the equinoctial, due to the action of the heavenly bodies on the protuberant ring of matter about the terrestrial equator, there is another effect due to the deflecting action of the planets. By this the earth is turned aside from the path it would describe, if subjected to the action of the sun alone, and the place of the ecliptic, therefore,

changed. The amount of this change is exceedingly small, being only about 46″ in a century. Its present effect is to diminish the mean obliquity, and this will continue to be the case for a long period of ages, when the change will be in the opposite direction, the motion being one of oscillation to the extent of 1° 21′ about a mean position. The change in the value of the mean obliquity arising from the cause here referred to, is called the *secular variation* of the obliquity, because of the great period of time required to pass through all its values.

SIDEREAL TIME.

§ 168. It has been explained (p. 237) how the motion of the pointers or hands of clocks and watches over stationary circular scales of equal parts upon their dial-plates, is employed to measure the lapse of time. The uniform motion of the meridian, carrying with it an imaginary movable circular scale of equal parts, coincident with the equinoctial, gives the means of regulating these and all other artificial time-keepers.

§ 169. The origin or zero of the equinoctial scale is on the upper meridian; its unit of measure is one hour, equal to 15°; its pointer or hand the declination circle through the centre of some heavenly body, and time measured upon it takes the name of the body which regulates the pointer.

§ 170. The distance of the pointer from the origin or upper meridian, estimated westwardly, is the hour angle of the body which gives the scale its name, and measures the time since its meridian passage.

§ 171. Time measured by the hour angle of the mean equinox is called *mean sidereal time;* and the interval of time between two consecutive passages of the meridian over the mean equinox, is called a *sidereal day.*

§ 172. Time measured by the hour angle of the apparent equinox is called *apparent sidereal time;* and the interval of time between two consecutive passages of the meridian over the apparent equinox, is called an *apparent sidereal day.*

§ 173. Apparent sidereal time is that usually employed by astronomers. It is affected by the equation of the equinoxes in right ascension, of which the value in time being applied to the apparent sidereal time, with its proper sign, gives the mean sidereal time. This difference between apparent and mean sidereal time is called also the *equation of sidereal time.*

§ 174. Apparent sidereal days are slightly unequal; but the fluctuations of a clock marking apparent from one noting mean sidereal time would be only about $2^s.3$ in nineteen years.

§ 175. A timepiece whose hour-hand passes uniformly over the circular

scale of 24 hours on the dial-plate, in a sidereal day, is said to run with sidereal time; it will mark mean sidereal time when its hands indicate at any and every instant the hour angle of the mean vernal equinox.

§ 176. The sidereal time of the meridian's passing the centre of any body is the true right ascension of the body; and the rate of the time-piece on sidereal time, its error at any epoch, and the indication of the hands on its dial-plate at the instant the meridian passes the centre of any body, are the data which make known the body's right ascension.

§ 177. The sidereal day is shorter than the time required for the earth to turn once about its axis by about $\frac{1}{120}$ of a sidereal second.

THE EARTH'S ORBIT.

§ 178. The orbit of the earth is an ellipse, of which the sun occupies one of the foci.

§ 179. The extremities of the transverse axis of the orbit are called the *Apsides;* that most remote from the sun is called the *higher* and that nearest to the sun the *lower apsis.* The lower apsis is also called the *perihelion* and the higher apsis the *aphelion.* The transverse axis produced both ways is called the *line of the apsides.*

§ 180. The place of the sun or other heavenly body which has the greatest distance from the earth is called the *apogee*, and that which has the least distance is called the *perigee.* When, therefore, the earth is in aphelion, the sun is in apogee; and when the earth is in perihelion, the sun is in perigee.

§ 181. The quotient obtained from dividing the circumference of a circle, of which the radius is unity, by the interval of time between two consecutive returns of a body to the same origin, is called the body's *mean motion* from that origin.

Thus, let T be the interval. and m the mean motion; then will

$$m = \frac{2\pi}{T} \quad \ldots\ldots\ldots\ldots \quad (43)$$

§ 182. The origin may be movable or fixed; when in motion, the motion may be direct or retrograde.

§ 183. Denote by r the radius vector of the earth, by c the area which this line describes in a unit of time, and by n the true motion, then will, Analytical Mechanics, equation (266),

$$n_{\prime} = \frac{2c}{r^2} \quad \ldots\ldots\ldots\ldots \quad (44)$$

§ 184. The interval of time between two consecutive returns of the sun to the vernal equinox, is called a *tropical year.* That between two consecutive returns to the mean vernal equinox, a *mean tropical year.*

§ 185. The arc of the ecliptic from the mean vernal equinox to the place the sun would occupy, had his motion in longitude been uniform and equal to a mean of his actual motions, is called his *mean longitude*—the true and mean places always coming together on the line of the apsides.

§ 186. The interval of time between two consecutive returns of the earth to the perihelion or aphelion is called an *anomalistic* year.

§ 187. The mean motion of the earth from perihelion is the value of m, in equation (43), the value of T therein being the anomalistic year.

§ 188. The angle $E\,S\,P$, which the radius vector of the earth makes at any time with the line of the apsides, reckoned from perihelion, is called the *true anomaly.*

§ 189. The angle which the radius vector of the earth at any time would make with the same line, and estimated from the same point, had the earth moved from perihelion with its mean motion, and retained this motion unaltered, is called the *mean anomaly.*

§ 190. The relation which connects the mean with the true anomaly is, Appendix No. V., equation (g),

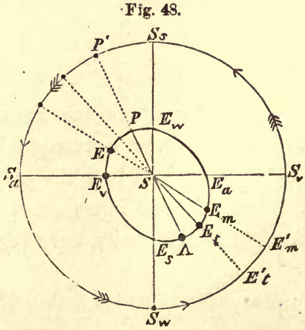
$$n = V - 2\,e \sin V + \tfrac{3}{4}\,e^2 \sin 2\,V - \text{\&c.} \quad . \quad . \quad . \quad (45)$$

in which n is the mean anomaly, V the true anomaly, and e the eccentricity.

§ 191. The difference between the mean and the true anomaly is called the *equation of the centre.* Denoting the equation of the centre by E, we have, equation (45),

$$E = n - V = -\,2\,e \sin V + \tfrac{3}{4}\,e^2 \sin 2\,V - \text{\&c.} \quad . \quad . \quad (46)$$

Fig. 48.



§ 192. Let $S_v\,S_s\,S_a\,S_w$ represent the ecliptic; $S_v\,S_a$ the line of the equinoxes; $S_s\,S_w$ the line of the solstices; S_v the vernal equinox; S the sun; $P\,E\,A\,E_a\,P$ the earth's orbit; P the perihelion; A the aphelion.

When the earth is at E_v the sun will appear at the vernal equinox S_v; when at E_s, the sun will appear at the summer solstice S_s; and when at E_a, the sun will

appear at the autumnal equinox S_a; and when at E_w, the sun will appear at the winter solstice S_w.

§ 193. Let E_t be the place of the earth, E_m its mean place; then will $S_v E'_t$, estimated in the order of the signs, that is, in the direction indicated in the figure, be the earth's longitude as seen from the sun; $S_v E'_m$, estimated in the same direction, its mean longitude; $S_v P'$ the longitude of the perihelion; $P'E'_t$ the true anomaly; $P'E'_m$ its mean anomaly, and $E'_t S E'_m$ the equation of the centre.

§ 194. It is obvious that the equation of the centre is equal to the difference between the mean and true longitudes from the same equinox.

§ 195. The earth's orbit is known when its *semi-transverse axis*, its *eccentricity*, and the *longitude of its perihelion* are known, its plane being that of the ecliptic. These are called the *elements of figure.* The *periodic time*, the *mean motion*, and the *mean longitude at some particular epoch*, are the additional data from which result by computation the earth's true motion and actual place at any other epoch before or after. These are called the elements of place and motion.

§ 196. Make

L = mean longitude of the earth at the given epoch;
t = an interval of time before or after;
m = mean motion;
α = true longitude at time of observation;
α_p = longitude of the perihelion:

then, Appendix No. V., equation (i),

$$L + m\,t = \alpha - 2\,e \sin(\alpha - \alpha_p) + \tfrac{3}{4}\,e^2 \sin 2\,(\alpha - \alpha_p) - \&c. \qquad (47)$$

The sun will have the greatest apparent diameter when the earth is in perihelion, and least when in aphelion; denote these diameters by $\delta_,$ and δ' respectively, and the corresponding radii vectors by $r_,$ and r'; then from the principles of optics,

$$r_, : r' :: \delta' : \delta_,,$$

and

$$r' - r_, : r' + r_, :: \delta_, - \delta' : \delta_, + \delta',$$

whence

$$\frac{r' - r_,}{r' + r_,} = e = \frac{\delta_, - \delta'}{\delta_, + \delta'}.$$

Actual measurements give about

$$\delta_, = 32',5$$
$$\delta' = 31'.5$$

whence
$$e = \frac{1}{64} = 0.016 \text{ nearly};$$

from which it appears that e is so small as to justify the omission from equation (47) of those terms in which its powers higher than the first enter, and we may write

$$L + m\,t = \alpha - 2\,e \sin(\alpha - \alpha_p) \quad . \quad . \quad . \quad . \quad . \quad (48)$$

§ 197. From four observed right ascensions of the sun, compute, by equation (33), his corresponding true longitudes; each longitude increased by 180° will give the corresponding true longitude of the earth; denote these by α_1, α_2, α_3, and α_4, and the intervals of time from the epoch of the mean longitude L, say noon, January 1st, to the times of observation, by t_1, t_2, t_3, and t_4 respectively, then will, equation (48),

$$\left.\begin{aligned} L + m\,t_1 &= \alpha_1 - 2\,e \sin(\alpha_1 - \alpha_p), \\ L + m\,t_2 &= \alpha_2 - 2\,e \sin(\alpha_2 - \alpha_p), \\ L + m\,t_3 &= \alpha_3 - 2\,e \sin(\alpha_3 - \alpha_p), \\ L + m\,t_4 &= \alpha_4 - 2\,e \sin(\alpha_4 - \alpha_p), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (49)$$

four equations, from which the mean longitude L at the epoch, the mean motion m, the eccentricity e, and longitude of the perihelion α_p, may be found. For this purpose subtract the first from the second,

$$m\,(t_2 - t_1) = \alpha_2 - \alpha_1 - 2\,e\,[\sin(\alpha_2 - \alpha_p) - \sin(\alpha_1 - \alpha_p)];$$

making
$$t_2 - t_1 = \theta, \quad \alpha_2 - \alpha_1 = a, \quad \text{or } \alpha_2 = \alpha_1 + a,$$

and reducing by the relation

$$\sin(\alpha_2 - \alpha_p) = \sin(\alpha_1 - \alpha_p + a) = \sin(\alpha_1 - \alpha_p)\cos a + \cos(\alpha_1 - \alpha_p)\sin a,$$

we find

$$m\,\theta = a + 2\,e\,[\sin(\alpha_1 - \alpha_p)(1 - \cos a) - \cos(\alpha_1 - \alpha_p)\sin a] \quad (50)$$

subtracting the first of equations (49) from the third and fourth, making

$$t_3 - t_1 = \theta', \quad t_4 - t_1 = \theta'';$$
$$\alpha_3 - \alpha_1 = a', \quad \alpha_4 - \alpha_1 = a'';$$

reducing in the same way, and replacing $1 - \cos a$ by its equal, $2 \sin^2 \frac{1}{2} a$, we find, including the equation above,

$$\begin{aligned} m\,\theta - a &= 2\,e\,[2 \sin(\alpha_1 - \alpha_p) \sin^2 \tfrac{1}{2} a - \cos(\alpha_1 - \alpha_p) \sin a], \\ m\,\theta' - a' &= 2\,e\,[2 \sin(\alpha_1 - \alpha_p) \sin^2 \tfrac{1}{2} a' - \cos(\alpha_1 - \alpha_p) \sin a'], \\ m\,\theta'' - a'' &= 2\,e\,[2 \sin(\alpha_1 - \alpha_p) \sin^2 \tfrac{1}{2} a'' - \cos(\alpha_1 - \alpha_p) \sin a'']. \end{aligned}$$

Dividing the first of these by the second and third successively, making

$$\left.\begin{aligned} \frac{m\,\theta - a}{m\,\theta' - a'} &= M, \\ \frac{m\,\theta - a}{m\,\theta'' - a''} &= N; \end{aligned}\right\} \quad \ldots \ldots \quad (51)$$

and dividing both numerator and denominator of the secord members by $\cos(\alpha_1 - \alpha_p)$, we have

$$\left.\begin{aligned} M &= \frac{2\tan(\alpha_1 - \alpha_p)\,.\,\sin^2\tfrac{1}{2}a - \sin a}{2\tan(\alpha_1 - \alpha_p)\,.\,\sin^2\tfrac{1}{2}a' - \sin a'}, \\ N &= \frac{2\tan(\alpha_1 - \alpha_p)\sin^2\tfrac{1}{2}a - \sin a}{2\tan(\alpha_1 - \alpha_p)\sin^2\tfrac{1}{2}a'' - \sin a''}; \end{aligned}\right\} \quad \ldots \quad (52)$$

from which we find

$$2\tan(\alpha_1 - \alpha_p) = \frac{M\sin a' - \sin a}{M\sin^2\tfrac{1}{2}a' - \sin^2\tfrac{1}{2}a} = \frac{N\sin a'' - \sin a}{N\sin^2\tfrac{1}{2}a'' - \sin^2\tfrac{1}{2}a} \quad (53)$$

in which the only unknown quantity is m; this entering, equations (51), the values of M and N.

To find the value of m, clear the fractions, transpose to the first member, and make

$$\begin{aligned} n &= \sin a \;\; \sin^2\tfrac{1}{2}a' - \sin a' \;\; \sin^2\tfrac{1}{2}a, \\ k &= \sin a' \;\; \sin^2\tfrac{1}{2}a'' - \sin a'' \;\; \sin^2\tfrac{1}{2}a', \\ i &= \sin a'' \sin^2\tfrac{1}{2}a \;\; - \sin a \;\; \sin^2\tfrac{1}{2}a'', \end{aligned}$$

or reducing for the sake of logarithmic computation by the relation

$$\sin a = 2\sin\tfrac{1}{2}a\,.\cos\tfrac{1}{2}a,$$

$$\left.\begin{aligned} n &= \;\;\, 2\sin\tfrac{1}{2}a \;\;\; \sin\tfrac{1}{2}a' \;.\sin\tfrac{1}{2}(a' - a\,), \\ k &= \;\;\, 2\sin\tfrac{1}{2}a'.\sin\tfrac{1}{2}a''.\sin\tfrac{1}{2}(a'' - a'), \\ i &= -2\sin\tfrac{1}{2}a \;\;\; \sin\tfrac{1}{2}a''.\sin\tfrac{1}{2}(a'' - a\,): \end{aligned}\right\} \quad \ldots \quad (54)$$

we have

$$M n + N i + k\,.\,M\,.\,N = 0.$$

Replacing M and N by their values given in equations (51), we find

$$\frac{n\,(m\,\theta - a)}{m\,\theta' - a'} + \frac{i\,(m\,\theta - a)}{m\,\theta'' - a''} + \frac{k\,(m\,\theta - a)^2}{(m\,\theta' - a')\,(m\,\theta'' - a'')} = 0\,;$$

whence

$$(m\,\theta - a)\,[\,(n\,\theta'' + i\,\theta' + k\,\theta)\,m - (n\,a'' + i\,a' + k\,a)\,] = 0.$$

But $m\,\theta - a$ cannot be zero, since e is not zero.

Placing, therefore, the second factor equal to zero, we find

$$m = \frac{n\,a'' + i\,a' + k\,a}{n\,\theta'' + i\,\theta' + k\,\theta} = \frac{a'' + \frac{i}{n}\,a' + \frac{k}{n}\,a}{\theta'' + \frac{i}{n}\,\theta' + \frac{k}{n}\,\theta} \quad . \quad . \quad . \quad (55)$$

From equations (54) we have

$$\left.\begin{aligned} \frac{i}{n} &= -\frac{\sin\frac{1}{2}\,a'' \,.\, \sin\frac{1}{2}\,(a'' - a)}{\sin\frac{1}{2}\,a' \,.\, \sin\frac{1}{2}\,(a' - a)}, \\ \frac{k}{n} &= \frac{\sin\frac{1}{2}\,a'' \,.\, \sin\frac{1}{2}\,(a'' - a')}{\sin\frac{1}{2}\,a \,.\, \sin\frac{1}{2}\,(a' - a)}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Now a, a', and a'' are the increments of the true longitude since the first observation; these in equations (56) give the fractions $\frac{i}{n}$ and $\frac{k}{n}$; these in equation (55) give the value of m; this in equations (51) and (52) give the value of tan $(\alpha_1 - \alpha_p)$ and therefore of α_p; this, in equation (50), gives the value of e, and this, together with m and α_p, in first of equations (49), gives the value of L.

§ 198. The mean motion in longitude, the eccentricity and longitude of the perihelion being determined at dates remote from one another, are found to be very slightly variable. The present value of the eccentricity is 0.01678356, the semi-transverse axis of the earth's orbit, or the earth's mean distance from the sun being unity; that of the mean motion in longitude in one sidereal day is 0°.98295603; the longitude of the perigee at the beginning of the present century was 279° 30′ 05″.0, and the mean longitude of the sun at the same time was 280° 39′ 10″.2.

The longitude of the perihelion is found to increase at a mean rate of 61″.9, in a tropical year, and deducting 50″.2 for the retrocession of the mean equinox, gives to the perihelion a direct motion of 11″.7 through space in the same time.

§ 199. Denoting by y_{t_*} the length of the tropical year in sidereal days, we have

$$y_{t_*} = \frac{360^\circ}{m} = \frac{360^\circ}{0^\circ.98295603} = 366.242 \text{ days} \quad . \quad . \quad . \quad (57)$$

MEAN SOLAR TIME.

§ 200. Although the mode of reckoning time by the motion of the vernal equinox affords great facilities in practical astronomy, it is of little or no use in the ordinary operations of common life. Business and social in-

tercourse are mostly regulated by the alternations of daylight and darkness, and the sun is the natural object of reference in all divisions of time for society in general.

§ 201. Time measured by the hour angle of the sun is *apparent solar time.*

§ 202. The epoch of the sun's being on the meridian of a place, is called *apparent noon* of that place.

§ 203. The interval of time between two consecutive passages of the sun's centre over the upper or lower meridian of the same place, is called an *apparent solar day.*

The apparent solar is longer than the sidereal day, in consequence of the earth's real, and therefore of the sun's apparent, motion in the ecliptic in an easterly direction. If, for instance, the vernal equinox and the sun were to pass the meridian of a place at the same instant to-day, the sun would be to the east of the equinox on the morrow, and would cross the same meridian after it.

§ 204. The orbital motion of the earth and, therefore, the apparent motion of the sun in the ecliptic is, Eq. (a), Appendix V, variable. The unequal arcs which measure the daily increments of the sun's longitude vary their inclination to the equinoctial from about 23° 28′ at the equinoxes, to zero at the solstices; and these unequal arcs may hence be projected by declination circles into still more unequal arcs of right ascension. These latter measure the excess of the different apparent solar over the sidereal days; and hence the variable orbital motion of the earth, and the inclination of the plane of its orbit to the equinoctial, conspire to make the lengths of the apparent solar days unequal.

§ 205. Timepieces cannot be made to imitate this inequality, nor is it desirable they should do so, were it possible.

Had the earth's orbit been circular and in the plane of the equinoctial, its orbital motion would have been uniform, the sun's apparent daily increase of right ascension constant, and the apparent solar days of equal duration.

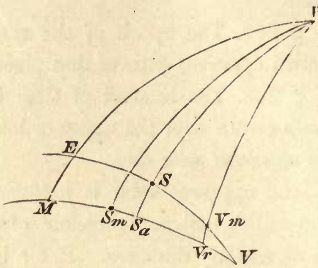
§ 206. These conditions are fulfilled by the device of an imaginary sun conceived to move uniformly in the equinoctial with the true sun's mean motion in longitude, and to set out from the reduced place of the mean vernal equinox when the true sun's mean place leaves the mean equinox.

This imaginary body is called the *mean sun.*

§ 207. Time measured by the hour angle of the mean sun is called *mean solar time.* The epoch of the mean sun being on the meridian of a place, is called *mean noon* of that place.

§ 208. The difference between the apparent and mean solar time is called the *equation of time.* If to the mean time the equation of time be applied with its proper sign, the apparent time will result; if the equation of time be applied with its proper sign to the apparent time, the mean time will result.

Fig. 49.



The equation of time is employed to pass from mean to apparent, or from apparent to mean time.

§ 209. Thus, let PM be an arc of the meridian, VM of the equinoctial, VE of the ecliptic; P the pole of the equinoctial; V the true, V_m the mean, and V_r the reduced place of the mean equinox; S the true and S_m the mean sun; then will MPS be apparent, and MPS_m mean solar time; VS_a the right ascension of the real sun; VV_r the equation of the equinoxes in right ascension.

Make

$e = S_a S_m =$ the equation of time;

$a = VS_a =$ the right ascension of true sun;

$l = V_r S_m =$ the mean longitude of the sun;

$q = VV_r =$ the equation of the equinoxes in right ascension;

then from the figure, we have

$$e = a - (l + q) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

that is, the equation of time is equal to the sun's true right ascension diminished by the sun's mean longitude, corrected for the equation of the equinoxes in right ascension.

§ 210. When the sun's true right ascension exceeds the corrected mean longitude, the equation of time must be added to apparent time to obtain mean time, and *vice versa.* The equation of time is zero four times a year, viz., on 15th April, 14th June, 31st August, and 24th December.

§ 211. The mean sun and mean equinox when together must pass some meridian at the same instant. When the same meridian returns to the mean equinox on the following day, the mean sun will be to the east by a distance equal to that which measures its motion in one sidereal day; and the mean solar day will exceed the sidereal day by the interval of sidereal time required for the meridian to overtake the mean sun after it passes the mean equinox.

Denote this excess by t, expressed in days; and the motion of the mean

sun in one sidereal day, equal to the earth's mean orbital motion in the same time, by m. Then will $m\,t$ be the motion of the mean sun in the time t, and its right ascension from the mean equinox at the instant the meridian overtakes it will be $m + m\,t$. But this is the hour angle of the mean equinox, or the sidereal time t, reduced to degrees; whence

$$m + m\,t = 360^\circ \times t;$$

or

$$t = \frac{m}{360^\circ - m};$$

and for the length of the mean solar day, expressed in sidereal time,

$$1 + t = 1 + \frac{m}{360^\circ - m};$$

or replacing m by its value $0^\circ.98295603$, § 198, and denoting the length of the mean solar day by D_m, expressed in terms of the sidereal day D_s, as unity, we have

$$D_m = 1.00273791\ D_s \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

and

$$D_s = \frac{D_m}{1.00273791} = 0.99726957\ D_m.$$

Whence to convert intervals of mean solar into intervals of sidereal, or intervals of sidereal into intervals of mean solar time, we have these rules, viz.:

$$\textit{Sidereal interval} = 1.00273791 \times \textit{Solar interval},$$

$$\textit{Solar interval} = 0.99726957 \times \textit{Sidereal interval}.$$

§ 212. Applying this second rule to the length of the tropical year expressed in sidereal days, we have, Eq. (59),

$$\textit{Solar interval} = 0.99726957 \times 366.242 = 365.2422414;$$

or reducing the fraction to hours, minutes, and seconds, and denoting the length of the tropical year, expressed in mean solar time, by y_{tm}, we have

$$y_{tm} = 365^d\ 5^h\ 48^m\ 48^s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

§ 213. Denote by y_{am} the length of the anomalistic year expressed in mean solar time; then, §§ 157 and 198,

$$360^\circ - 50''.2 : 360^\circ + 11''.7 :: 365^d\ 5^h\ 48^m\ 48^s : y_{am};$$

whence

$$y_{am} = 365^d\ 6^h\ 13^m.3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

§ 214. The interval of time required for the earth to perform one entire circuit about the sun in space is called a *sidereal year*.

Denote by y_{sm} the length of the sidereal year in mean solar time, then

$$360^\circ - 50''.2 : 360^\circ :: 365^d\ 5^h\ 48^m\ 48^s : y_{sm};$$

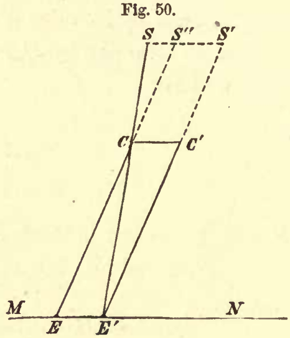
whence

$$y_{sm} = 365^d\ 6^h\ 9^m\ 9^s.6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

ABERRATION.

§ 215. The earth's orbital motion, combined with the motion of light, produces an apparent displacement of all the heavenly bodies in the direction of the point of the celestial sphere towards which the earth is, at the instant, moving. This displacement is called *aberration.*

Fig. 50.



Thus, let S be the place of a heavenly body, E that of the earth moving from M towards N along an arc of its orbit. From E take any distance EE'; join S and E', and lay off upon $E'S$ the distance $E'C$, which bears to EE', the ratio of the velocity of light to that of the spectator, and suppose C connected like himself with the earth.

Now a wave of light from S, another which originates at C, when that from S passes this point, and the spectator's eye starting from E at the same instant, will all meet at E', and as bodies always appear in the direction of the normal to the wave front, the point C and the body S will be seen in the direction $E'S$. But C, having a velocity equal and parallel to the spectator, will have passed on to C', at the extremity of a line through C equal and parallel to EE'; so that when C appears in the direction of the body S, it will, in fact, be in advance of it by the angle $SE'C'$.

Let C be the optical centre of the object-glass of a telescope, attached to the face of a graduated circle, moving in the plane of the body and the tangent line to the terrestrial orbit at the earth's place, and E' the intersection of the cross wires at the solar focus; then, when the image of the body appears at the latter point, the line of collimation will be in advance of the body itself, and its instrumental bearing will be in error by the angle $SE'C'$, and must be corrected by the same angle to get the true bearing.

§ 216. But had S been a terrestrial object, by the time its light from

the position S had reached C, the body itself would have been at S'', the intersection of $E\,C$ produced and $S\,S'$ drawn parallel to $E\,E'$; and at the instant of its light reaching E' the body would have been at S', the intersection of $S\,S''$ produced and the line of collimation. Geodetical observations are, therefore, unaffected by aberration, while astronomical observations are, in general, affected by it.

§ 217. Make

$r = S\,E'\,S' = E\,C\,E' =$ aberration;

$\alpha = S\,E'\,N =$ angle the direction of the body makes with that of the earth's motion.

$V =$ velocity of the earth;

$V' =$ velocity of light:

Then, in the triangle $C\,E'\,E$,

$$V' : V :: \sin(\alpha - r) : \sin r,$$

whence

$$\sin r = \frac{V}{V'} \cdot \sin(\alpha - r) \quad . \; . \; . \; . \; . \; . \; . \quad (64)$$

If $\rho_{\prime}$ denote the mean radius of the earth's orbit, then will

$$V = \frac{2\,\pi\,\rho_{\prime}}{365^{d}.25636};$$

and it will be shown hereafter that light requires $16^{m}\ 26^{s}$ to pass over the distance $2\,\rho_{\prime}$, and therefore

$$V' = \frac{2\,\rho_{\prime}}{16^{m}\ 26^{s}};$$

whence

$$\frac{V}{V'} = \frac{3.1416 \times 16^{m}\ 26^{s}}{365^{d}.25636} = 0.00009815,$$

from which, and equation (64), it is apparent that r is very small, and may be neglected in comparison with α; we may therefore write

$$\frac{r''}{206264''.8} = 0.00009815 \sin \alpha,$$

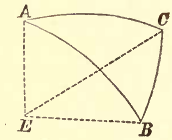
in which 206264.8 is the number of seconds in radius; whence

$$r'' = 0.00009815 \times 206264''.8 \sin \alpha,$$

or

$$r'' = 20''.246 \sin \alpha \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (65)$$

Fig. 51.



§ 218. Let AB be the intersection of the celestial sphere by a plane through the body and the direction of the earth's motion, AC that of a plane through the observer and star, and perpendicular to the plane of the ecliptic, and BC an arc of the ecliptic; then will B be the point in which the tangent to the earth's orbit at the place of the earth pierces the celestial sphere, A will be the projection of the body upon the celestial sphere, and $AB = \alpha$; and if $AC = \lambda$ and $CAB = \varphi$, we have

$$\cos \varphi = \tan \lambda \cot \alpha,$$

and

$$\cos^2 \varphi = \tan^2 \lambda . \cot^2 \alpha,$$

whence

$$1 - \sin^2 \varphi = \tan^2 \lambda . \frac{1 - \sin^2 \alpha}{\sin^2 \alpha},$$

and solving with respect to $\sin \alpha$,

$$\sin \alpha = \frac{\tan \lambda}{\sqrt{1 + \tan^2 \lambda - \sin^2 \varphi}} = \frac{\tan \lambda}{\sqrt{\sec^2 \lambda - \sin^2 \varphi}},$$

and therefore,

$$\sin \alpha = \frac{\sin \lambda}{\sqrt{1 - \cos^2 \lambda . \sin^2 \varphi}};$$

and this in equation (65) gives

$$r = \frac{20''.246 \sin \lambda}{\sqrt{1 - \cos^2 \lambda . \sin^2 \varphi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

which is the polar equation of an ellipse, the pole being at the centre. So that, if the image of a fixed star were kept constantly on the cross wires of a telescope during one entire revolution of the earth in its orbit, the line of collimation would trace upon the celestial sphere an ellipse of which the star would occupy the centre; the semi-transverse axis would be $20''.246$ and the eccentricity $\cos \lambda$.

If the star were in the plane of the ecliptic, then would $\lambda = 0$, $\cos \lambda = 1$, and the orbit would become a right line equal in length to $40''.492$. If the star were at either pole of the ecliptic, then would $\lambda = 90$, $\cos \lambda = 0$, and the orbit would be a circle. Between these limits the eccentricity will vary from 1 to 0.

The coefficient $20''.246$ is called the *constant of aberration.*

§ 219. Since the aberration is in the arc AB, its projection on AC will be the aberration in latitude. Denoting the latter by r', we have

$$r' = \frac{20''.246 \,.\, \sin \lambda \,.\, \cos \varphi}{\sqrt{1 - \cos^2 \lambda \,.\, \sin^2 \varphi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

which is obviously the greatest when $\varphi = 0°$ or $180°$, in which case the earth will be moving parallel to the circle of latitude of the body, and the aberration in latitude will be equal to $20''.246 \cdot \sin \lambda$, which is the semi-conjugate axis of the ellipse.

The aberration in longitude denoted by $r_{\prime}$ will give

$$r_{\prime} = \frac{20''.246 \,.\, \tan \lambda \,.\, \sin \varphi}{\sqrt{1 - \cos^2 \lambda \,.\, \sin^2 \varphi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

which is the greatest when $\varphi = 90°$ or $270°$, in which case the earth will be in the act of passing the body's circle of latitude, and the corresponding aberration will be $20''.246$, the semi-transverse axis of the ellipse.

§ 220. Equations (66), (67), and (68) are applicable to a body which has no proper motion of its own. In case the body has a motion, this must be allowed for in clearing its instrumental bearing of aberration, and the mode of doing this will be indicated under the head of planets.

§ 221. In the case of the sun, which may be regarded as fixed, φ is always $90°$, $\sin \varphi = 1$, and replacing $1 - \cos^2 \lambda$ by $\sin^2 \lambda$, equation (66), reduces to

$$r = 20''.246\,;$$

that is, the sun will always appear behind his true place by the constant of aberration.

§ 222. In conclusion, it is proper to remark that V, the velocity of the earth in its orbit, which is assumed to be constant, is not strictly so, but the variation is so small as not sensibly to affect the foregoing results. The actual velocity varies inversely as the length of the perpendicular drawn from the sun to the line which is tangent to the earth's orbit at the earth's place (*Analyt. Mechanics*, § 193). But the eccentricity of the orbit being very small, gives but little variation in this perpendicular.

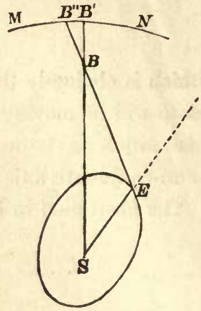
HELIOCENTRIC PARALLAX.

§ 223. The place in which a body would appear if viewed from the centre of the sun, is called its *Heliocentric place.*

§ 224. The arc of a great circle of the celestial sphere drawn from the heliocentric to the geocentric place of a body, is called its *Heliocentric parallax;* and is obviously the path a body would appear to describe to

an observer were he to pass from one extremity to the other of the earth's radius vector.

Fig. 52.



Thus, let S be the sun, E the earth, B the body, and MN the intersection of the celestial sphere by a plane through the body and radius vector SE; then will B' be the heliocentric, and B'' the geocentric place of the body; and $B''B'$ its heliocentric parallax. The heliocentric parallax measures the angle $B''BB' = SBE =$ *the angle at the body subtended by the radius vector of the earth.*

§ 225. Make

$$D = SB \quad = \text{distance of body from sun};$$
$$R = SE \quad = \text{earth's radius vector};$$
$$r_{,} = SBE = \text{heliocentric parallax};$$
$$\alpha_{,} = SEB = \text{angular distance between sun and body};$$

then in the triangle SEB,

$$D : R :: \sin \alpha_{,} : \sin r_{,},$$

whence

$$\sin r_{,} = \frac{R}{D} . \sin \alpha_{,} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

When $\alpha_{,} = 90°$, then will $r_{,}$ be the greatest possible. This maximum heliocentric parallax, is called the *annual parallax*; which denote by π, and we have

$$\sin \pi = \frac{R}{D};$$

and if π be very small,

$$\frac{\pi}{\omega} = \frac{R}{D},$$

π is expressed in seconds, and ω denotes the number of seconds in radius. From this we obtain

$$D = R . \frac{\omega}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

This gives the distance of the body from the sun in terms of its annual parallax and the earth's radius vector.

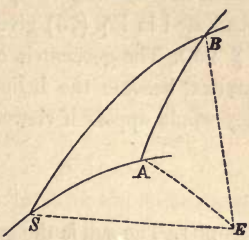
§ 226. Substituting the value of D in Eq. (69), and making

$$\sin r_{,} = \frac{r_{,}}{\omega},$$

we have

$$r_{,} = \pi . \sin \alpha_{,} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Fig. 58.



§ 227. Let S be the sun's place in the ecliptic, B the place of the body, BA the arc of a circle of latitude, SA an arc of the ecliptic, and E the earth. The side SB will measure the angle $\alpha_{,}$; and denoting the side AB by λ, and the angle SBA by $\varphi_{,}$, we have

$$\cos \varphi_{,} = \tan \lambda \,.\, \cot \alpha_{,};$$

and by a transformation, the same as in § 218,

$$\sin \alpha_{,} = \frac{\sin \lambda}{\sqrt{1 - \cos^2 \lambda \,.\, \sin^2 \varphi_{,}}};$$

which in Eq. (71) gives

$$r_{,}'' = \frac{\pi \,.\, \sin \lambda}{\sqrt{1 - \cos^2 \lambda \,.\, \sin^2 \varphi_{,}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

This is the polar equation of an ellipse having the pole at the centre; and it shows that the parallactic path of a body's geocentric place, due to the earth's orbital motion alone, is an ellipse of which the centre is the body's heliocentric place.

The semi-transverse axis and eccentricity are respectively π and $\cos \lambda$. If the body be in the pole of the ecliptic, then will $\lambda = 90$, $\cos \lambda = 0$, and the ellipse becomes a circle; if in the ecliptic, then will $\lambda = 0$, $\cos \lambda = 1$, and the ellipse becomes a right line whose length is $2\,\pi$.

§ 228. Heliocentric parallax throws a body from its heliocentric place towards the geocentric place of the sun or towards that point in which the earth's radius vector, produced beyond the sun, pierces the celestial sphere. Aberration throws it towards the point in which the tangent line to the earth's orbit, at the place of the earth, pierces the same surface. Both points are in the ecliptic, and if we neglect the eccentricity of the earth's orbit, which we may do without sensible error when the heliocentric parallaxes are employed, these points are 90° apart. When, therefore, $\varphi = 0°$ in Eq. (66), then will $\varphi_{,} = 90°$ in Eq. (72), and *vice versa ;* and the least possible heliocentric parallax will occur at the time of the greatest aberration, and the least aberration at the time of the annual parallax.

§ 229. When the longitudes of the sun and body differ by 90° or 270°, the heliocentric parallax will become the annual; and if the longitudes and latitudes of the body be taken at these times and cleared from the effects of aberration and nutation, there will result the longitudes and latitudes of two points separated by $2\,\pi$ or double the annual parallax. The value of

π then becomes known by a simple proposition in spherical geometry, and substituted in Eq. (70) gives the body's distance from the sun.

§ 230. The geocentric co-ordinates of a body corrected for heliocentric parallax, become the heliocentric co-ordinates, that is, the co-ordinates as they would appear if viewed from the centre of the sun.

THE SEASONS.

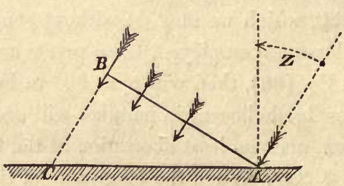
§ 231. The sun is the great fountain of those ethereal undulations which, acting upon the material of the earth's crust, give to the latter its surface heat; and the temperature of a place depends upon its exposure to their calorific action. While the sun is above the horizon, the place is receiving heat, and while below, parting with it; and in such proportion that the whole quantity gained and lost balance each other, since every location has nearly a constant average of annual mean temperature, as indicated by the thermometer.

§ 232. Whenever the sun is above the horizon more and beneath less than twelve hours, the general temperature of the place will be above the average, and the converse.

§ 233. A portion of the wave having a front surface equal to unity can generate but a limited quantity of heat, and, all other things being equal, the temperature at any one location will be inversely proportional to the extent of the earth's surface upon which this unit is made to act. If the wave front be parallel to the earth's surface the temperature will be greatest, for then the action is confined to the narrowest limits; if very oblique, the temperature will be low because the action is diffused over a larger space.

§ 234. Let AB be the section, by a vertical plane through the sun's centre, of a portion of the wave front, the surface of this portion being unity, say ten square miles; and AC the projection of the same on the earth's surface by normals to the wave front, called rays.

Fig. 54.



The sections are sensibly rectilinear within the limits assumed, and the rays being normal to the wave front, make with the line of the zenith and nadir to the earth's surface, an angle equal to BAC, equal to the sun's zenith distance, which being denoted by z, we have

$$AC = AB \,.\, \sec z.$$

Denote by I' the temperature when the wave and earth surfaces are parallel, and by I when they are oblique; then

$$AB \,.\, \sec z \,:\, AB \,::\, I' \,:\, I;$$

whence

$$I = \frac{I'}{\sec z} = I' \,.\, \cos z;$$

and if $I_{,}$ denote the temperature which would result at the unit's distance from the sun, and r the radius vector of the earth, we have from the law of diffusion, depending upon distance,

$$I' = \frac{I_{,}}{r^2};$$

whence

$$I = \frac{I_{,}}{r^2} \,.\, \cos z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

§ 235. Resuming Eq. (6), and making $p = 90° - d$, in which d denotes the sun's declination, we have

$$\cos z = \sin l \,.\, \sin d + \cos l \,.\, \cos d \,.\, \cos P \quad . \quad . \quad . \quad (74)$$

which, in Eq. (73), gives

$$I = \frac{I_{,}}{r^2} \,.\, [\sin l \,.\, \sin d + \cos l \,.\, \cos d \,.\, \cos P] \quad . \quad . \quad (75)$$

This result is wholly independent of terrestrial longitude, and is only dependent on the latitude of the place, the sun's declination, and the place of the earth in its orbit. All places upon the same parallel are equally exposed, therefore, to the solar influence, and whatever differences of mean temperature and of climate they may exhibit are due to local causes, such as the vicinity of mountains, extended plains, forests, deserts, or large bodies of water, upon all of which the sun is known to produce great variety of thermal effects.

§ 236. Making $z = 90°$, in Eq. (74), we have

$$\cos P = - \tan l \,.\, \tan d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

and making $P = 0$, in Eq. (75), we have

$$I = \frac{I_{,}}{r^2} \cos (l - d) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

Eq. (76) gives the value of the semi-upper diurnal arc, or the time the sun is above the horizon, or the duration of calorific action; and Eq. (77) the intensity of the solar influence when greatest.

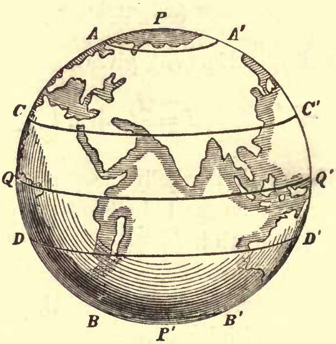
§ 237. In the course of the tropical year the declination varies nearly 47°, the sun being at one time about 23°.5 north, and at another about the same distance south of the equator.

As long as the latitude and declination are of the same name, that is, both north or both south, the sun will, Eq. (76), be longer than twelve hours above the horizon, and the place will receive more heat than it loses. And in proportion as the latitude and declination approach to equality, the intensity of the solar action will, Eq. (77), approach its maximum. This periodical variation in the daily average temperature of a place, caused by a change of the sun's declination, gives rise to the phenomena of the *seasons.*

§ 238. The interval of time during which the daily increment of temperature of a place is *increasing* is called its *spring;* that during which this increment is *decreasing* is called its *summer;* that during which the daily decrement is *increasing* is called its *autumn* or *fall;* and that during which this decrement is *decreasing* is called its *winter.*

§ 239. Within the tropics CC' and DD', and especially about the equator QQ', the temperature is, Eqs. (76) and (77), nearly uniform, and always high. On this account the terrestrial belt bounded by the tropics is called the *torrid zone.*

Fig. 55.



Between the tropics and polar circles AA' and BB' the average daily temperature is much less uniform and always lower than in the torrid zone. The belts bounded by the tropics and polar circles are called *temperate zones.*

Between the poles P and P' and polar circles, the variation of the average daily temperature is the greatest possible and the temperature itself least. The portions of the earth's surface about the poles and bounded by the polar circles are called *frigid zones.*

§ 240. Places within the torrid zone may be said to have two of each of the seasons during a tropical year, and all places in the temperate and frigid zones but one.

For all places in the north temperate and frigid zones, spring begins when the sun is on the equator and passing from south to north, or on the 20th March; summer, when the sun reaches the tropic of Cancer, or on the 21st June; autumn, when the sun returns to the equator in passing to

the south, or 22d September; and winter, when the sun reaches the tropic of Capricorn, or 21st December. For all places in the south temperate and frigid zones the names of the seasons will be reversed—spring becomes autumn, and summer winter.

§ 241. The elliptic form of the earth's orbit causes the radius vector, and therefore, Eq. (77), the intensity of the solar heat, to vary. But the angular velocity of the earth about the sun also varies, and according to the same law, viz.: that of the inverse square of the earth's distance from the sun—*Analytical Mechanics*, Eq. (266). Equal amounts of heat will therefore be developed while the earth is describing equal arcs of longitude, and the supply will be the same during the description of any two segments, equal or unequal, into which the entire orbit is divided by a line through the sun. The earth is nearer the sun while the latter is south of the equinoctial, or from the latter part of September to the latter part of March; and it describes the corresponding part of its orbit in a time so much shortened as just to balance the increase of thermal intensity. But for this law of compensation, the effect would be to increase the difference of summer and winter temperature in the southern and to diminish it in the northern hemisphere. As it is, however, no such inequality is found to subsist, but an equal and impartial distribution of heat and light is accorded to both hemispheres.

§ 242. But it must not be inferred that the mean surface heat is constant throughout the year; for such is not the fact. By taking, at all seasons, the mean of the temperatures of places diametrically opposite to one another, Professor Dove finds the mean temperature of the whole earth's surface in June considerably greater than that in December. This is due to the greater amount of land in that hemisphere which has its summer solstice in June; the thermal effect of the sun on land being greater than that on water.

§ 243. The variation of the radius vector amounts to about $\frac{1}{30}$ of its mean value, and therefore the fluctuation of heat intensity to about $\frac{1}{15}$ of its average measure—a circumstance which is manifested in a great excess of local heat in the interior of Australia during a southern, over that of the deserts of Africa during a northern summer.

TRADE WINDS.

§ 244. A discussion of the *trade winds*, the *earth's magnetism*, and the *tides*, belongs, in strictness, rather to terrestrial physics than to astronomy; but the necessary connection of these phenomena with the earth's diurnal

rotation and the action of foreign bodies upon the earth, as well as their importance to navigation, make a sufficient apology for introducing them here.

§ 245. The surface of the torrid zone is most heated; its excess of temperature is communicated to the superincumbent atmosphere; the latter is expanded, and becoming specifically lighter, is pressed upward by the colder portions on the north and south which move in and take its place. These, in their turn, are heated, expanded, and pressed upward, and a constantly ascending current is thus produced over an entire zone, of which the boundaries fluctuate with the varying declination of the sun and the proportion of land and water on the belt of the earth's crust lying immediately under the sun's diurnal path. The air thus accumulated at the summit of the ascending column, being unsupported on the north and south, flows off under the action of its own weight in either direction towards the poles, and, after cooling, descends again to the earth's surface in the higher latitudes of the temperate zones to supply the place and follow the course of that which has passed to the torrid zone.

§ 246. Two atmospheric rings, as it were, distinguished by peculiarities of internal circulation, are thus made to belt the earth on either side of the equator in directions parallel or nearly so to that great circle. On the lower side of these rings, in contact with the earth, the air moves towards the base of the ascending column, and on the upper towards the poles.

Fig. 56.

§ 247. By the diurnal motion of the earth, places on the equator have the greatest velocity of rotation, and all other places less in the proportion of the radii of their respective parallels of latitude. The portions of the ascending column which flow towards the poles set out with the eastward intertropical velocity, which they carry with them in part to the higher latitudes, where they descend to the earth's surface. To an observer situated in these latitudes, the air will have an apparent eastwardly motion, approaching to the excess of the intertropical velocity over that of the observer's parallel. Here *westerly winds* prevail.

§ 248. On parallels a few degrees lower, the tendency of the air is

towards the equator, and this combined with what remains of the apparent easterly component, just referred to, gives rise in the northern hemisphere to a *northwesterly* and in the southern to a *southwesterly* wind.

§ 249. In its onward course towards the equator, this same air crosses successively parallels of greater and greater velocity, and this, together with friction against the earth's surface, reduces the air's excess of easterly motion to zero, and here *northerly* winds prevail in the northern and *southerly* winds in the southern hemisphere.

§ 250. In latitudes still lower, the excess of rotation is in favor of the earth's surface, and the air, unable to keep up, now lags behind, and apparently tends to the west; and here, if the places be in the northern hemisphere, *northeasterly*, and if in the southern hemisphere *southeasterly* winds prevail.

§ 251. Nearer to the equator the radii of the parallels vary less rapidly, and the velocities of places on the same meridian are more nearly equal. In crossing these parallels the air in its onward course finds less variation in the velocity of the earth's surface, and friction, which now urges the air to the east, together with the easterly pressure below, arising from the westerly lagging in the summit of the ascending column, due to its decreasing angular motion as it recedes from the centre of rotation, soon brings the air and earth to relative rest. This occurs within the base of the ascending column where the currents of air, which are continually approaching each other from the directions of the poles, meet. This is, therefore, a region of *calms*.

§ 252. The aerial currents thus produced under the combined influence of solar heat and the diurnal motion of the earth, are called *Trade winds;* and they are so called from the benefits they are continually conferring on trade dependent upon navigation.

§ 253. A voyage from the United States to northern Europe in a sailing vessel is on an average ten days shorter than in the contrary direction. A sailing vessel on a passage from northern Europe to the southern coast of the United States would proceed to the Madeiras to take the easterly trades, and returning would proceed to the Bermudas to catch westerly trades.

§ 254. Within the region of calms the ascending column of air carries with it a large amount of aqueous vapor. In its ascent the air expands, its temperature is depressed, its aqueous vapor is first condensed into clouds, then into rain, and thus the region of calms is also a region of dense clouds and copious rains; the former giving to the earth, as viewed from

a distance, the appearance of being girted by dark broken belts, arranged in zones parallel to the equator.

§ 255. The limits of the trades do not always occur in the same latitudes, but vary with the season. In December and January, when the sun is furthest south, the northern boundary of the northeast trades of the Atlantic is about 20° N., whilst in the opposite season, from June to September, it is 32° N.

§ 256. Owing to the great disparity in the effects of solar heat upon land and water, and to the influence of mountain ranges and valleys upon atmospheric currents, the regular trades only occur, as a general rule, at sea, though in some level countries, within or near the tropics, constant easterly winds prevail. This is remarkably the case over the vast plains drained by the Amazon and lower Orinoco.

§ 257. The trades of the ocean and of the land are separated by a belt, within which other and variable winds occur. This belt lies upon the ocean, and extends along the coasts. When to the east of the trades, it is often a hundred miles wide, but when to the west its width is much smaller. The interruption of the trades, here referred to, is due to the difference of temperature of the air on sea and land, which changes with the seasons. The air over the land in the higher latitudes is the warmer when the meridian zenith distance of the sun is least, and colder when greatest. During the first period the wind is from the sea to the land, and in the second from the land to the sea, thus giving rise to the periodical winds called *Monsoons*, which occur even within the limits of the trades. A large island thus circumstanced is surrounded by a wind blowing from all quarters at the same time.

§ 258. A similar difference of temperature, but which varies with the alternations of day and night, gives rise to what are called the *sea* and *land breezes.*

TERRESTRIAL MAGNETISM.

§ 259. Another most important effect from the solar heat, combined with the diurnal motion of the earth, is the *earth's magnetism.*

§ 260. A difference of temperature in different parts of any body forming a continuous circuit is ever accompanied by electrical waves, propagated from the hotter to the colder parts. If the circuit be composed of various materials, possessing different powers of conducting heat, this difference may be maintained in greater degree and duration, and the effects of the electrical flow rendered more strikingly manifest.

§ 261. When the source of heat is moved gradually along the circuit, the electrical flow is in the direction of this motion, the colder portions always lying in advance and the warmer behind the moving source.

§ 262. A compass-needle, brought within the influence of such a circuit, will arrange itself at right angles to the direction of the flow, and under the same circumstances the same end of the needle will always point in the same direction. All this is the result of observation and experiment.

§ 263. The earth's crust is one vast thermo-electrical circuit, and its source of heat is the sun.

§ 264. In the diurnal motion of the earth, the different portions of its tropical regions are heated in succession by the sun during the day, and cooled by radiation during the succeeding night. The hotter portions will therefore lie to the east and the colder to the west of the sun's place. A perpetual flow of electricity is thus developed and maintained in and about the earth's crust from east to west, and gives rise to the earth's magnetic action.

§ 265. Were the materials of the earth all equally good electrical conductors, and the sun always in the equinoctial, the electrical flow would be parallel to that great circle, and the compass-needle would always point directly north and south. But neither of these conditions obtains. The materials vary greatly in conducting power, and the sun's declination is ever changing.

§ 266. The disparity of conducting power directs the electrical flow in paths of double curvature, of which the general direction is parallel to the equator, and the varying declinations of the sun are perpetually shifting their precise location and shape as well as changing the intensity of the flow.

§ 267. The position of stable equilibrium, assumed by a magnetic needle reduced to its axis, freely suspended from its centre of gravity, and subjected alone to the directive action of the earth's magnetism, is called the *magnetic position* of the place.

§ 268. The intersection by a vertical plane through the magnetic position with the celestial sphere, is called the *magnetic meridian.*

§ 269. The angle made by the magnetic and the true meridian is called the *magnetic declination*, or simply *declination.*

§ 270. The inclination of the magnetic position to the horizon is called the *magnetic inclination* or *dip.*

§ 271. The magnetic position at the same place is continually varying. It describes daily a conical surface, of which the place is the vertex, and

daily mean position the axis, while this axis itself describes a similar surface once a year about an *annual* mean position.

§ 272. The mean of all the declinations and of dips throughout any one day are the declination and dip for that day, and are called the *diurnal declination and dip.* The mean of all the diurnal declinations and dips for the different days throughout any given year, are the declination and dip for that year, and are called the *annual declination and dip.*

§ 273. The daily and annual fluctuations here referred to are called *periodic changes.* The annual declination and dip also change, and these changes, which are found to take place in the same direction for a great many years, are called *secular changes.*

§ 274. The magnetic declination and dip vary, in general, with the locality. The line connecting those places where the declination is zero, is called the *line of no declination;* and the line through the places where the dip is zero, is called the *magnetic equator.*

Fig. 57.

§ 275. According to the Magnetic Atlas of Hansteen, constructed for 1787, the line of no declination is found on the parallel of 60° north, a little to the west of Hudson's Bay; it proceeds in a southeasterly direction, through British America, the northwestern lakes, the United States, and enters the Atlantic Ocean near Chesapeake Bay, passes near the Antilles and Cape St. Roque, and continues on through the southern Atlantic till it cuts the meridian of Greenwich in south latitude 65°. It reappears in latitude 60° south, below New Holland, crosses that island through its centre, runs up through the Indian Archipelago with a double sinuosity, and crosses the equator three times—first to the east of Borneo, then between Sumatra and Borneo, and again south of Ceylon, from which it passes to the east through the Yellow Sea. It then stretches across the

coast of China, making a semicircular sweep to the west till it reaches the parallel of 71° north, when it descends again to the south, and returns northward with a great semicircular bend, which terminates in the White Sea.

On the magnetic chart this line is accompanied through all its windings by other lines upon which the declination is 5°, 10°, 15°, &c.; the latter becoming more irregular as they recede from the line of no declination. The use of these lines is to point out to navigators sailing by compass, the bearing of the true meridian from the magnetic.

§ 276. On the east of the American and west of the Asiatic branch of the line of no declination, the declination is west, while to the west of the American and east of the Asiatic branch the declination is east.

§ 277. The magnetic equator cuts the terrestrial equator, according to Hansteen, in four, and to Morlet in two points, called *nodes*, one of which is in the centre of Africa.

§ 278. Beginning at the African node the magnetic equator advances rapidly to the north, and quits Africa a little south of Cape Guardafui, and attains its greatest north latitude, 12°, in 62° of east longitude from Greenwich. Between this meridian and 174° east, the magnetic is constantly to the north of the terrestrial equator. It cuts the Indian peninsula a little to the north of Cape Comorin, traverses the Gulf of Bengal, making a slight advance to the terrestrial equator, from which it is only 8° distant a its entrance into the Gulf of Siam. It here turns again a little to the north, almost touches the north point of Borneo, traverses the straits between the Philippines and the isle of Mindanao, and on the meridian of Naigion it again reaches the north latitude of 9°. From this point it traverses the archipelago of the Caroline Islands, and descends rapidly to the terrestrial equator, which it cuts, according to Morlet in 174°, and according to Hansteen in 187° east longitude. Its next point of contact with the equator is in west longitude 120°. Here, according to Morlet, it does not pass into the northern hemisphere, but bends again to the south, while Hansteen makes it cross to the north, and continue there for a distance of 15° of longitude, and then return southward and enter the southern hemisphere in longitude 108° west, or 23° from the west coast of America. Between this point and its intersection with the terrestrial equator in Africa, the magnetic equator lies wholly in the southern hemisphere, its greatest southern latitude being about 25°.

§ 279. The dip increases as the needle recedes on either side from the magnetic equator, the end of the needle which was uppermost in the northern being lowermost in the southern hemisphere.

§ 280. The points at which the magnetic needle is vertical are called the *magnetic poles.* Of these there are four, two in each hemisphere, their positions being indicated on the magnetic charts.

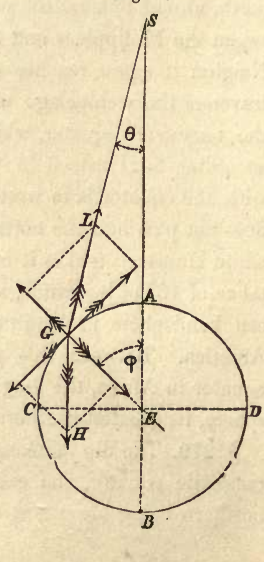
§ 281. On the magnetic charts, the magnetic equator is accompanied by curves of equal dip as in the case of the lines of equal declination.

§ 282. The line of no declination and the nodes of the magnetic equator are found to have a slow westerly motion, thus causing the different lines of equal declination and dip to pass successively through the same place, and illustrating the utter worthlessness of all maps constructed from compass bearings unless the diurnal declinations of the needle are carefully ascertained and recorded thereon.

§ 283. The intensity of the earth's magnetic action increases with the proximity of the electrical paths to the needle and with the difference of temperature in their different parts; and from changes in these, produced by the varying zenith distance of the sun during the day, and of his meridian zenith distance throughout the year, arise the daily and annual mutations of declination and dip; while to changes of the earth's crust, produced by geological causes, and increased cultivation of the soil from the spread of civilization, are to be attributed the secular variations of the same elements.

TIDES.

Fig. 58.



§ 284. Those periodical elevations and depressions of the ocean by which its waters are made to flow back and forth through the estuaries that indent our coasts, are called *Tides.*

§ 285. Perpetual change in the weight of the waters of the ocean, due to the attraction of the sun and moon upon the earth, and the diurnal rotation of the latter about its axis, cause and maintain the tides.

§ 286. Let $ACBD$ be a great circle of the earth, in a plane through the sun's centre at S. Draw SE through the earth's centre at E, and CD through the same point, and at right angles to SE. Assume any unit of mass as that at G; join G and S, and make

$d = SE \quad$ = distance of sun from the earth;
$\rho = EG \quad$ = radius of the earth;
$z = SG \quad$ = distance of G from sun;
$\varphi = AEG$ = angular distance of G from sun;
$\theta = GSE$ = angle at sun subtended by radius ρ;
m = mass of sun;
k = the attraction of unit of mass at unit's distance.

Then, since the attraction on unit of mass is proportional to the attracting mass directly, and the square of the distance inversely, the sun's action on G will be

$$\frac{k\,m}{z^2};$$

or because

$$z^2 = d^2 + \rho^2 - 2\,d\,\rho\cos\varphi,$$

$$\frac{k\,m}{d^2 + \rho^2 - 2\,d\,\rho\cos\varphi}.$$

But each unit of the earth's mass is acted upon by a centrifugal force equal and contrary to the centripetal force impressed upon the unit of mass to deflect it from its tangential into its orbital path. This latter is, by making $\rho = 0$, in the above

$$\frac{k\,m}{d^2},$$

and applying this to G in the direction GH parallel to SE, we have all the action on G arising from the sun's attraction.

Resolving these forces into their components in the direction of the radius EG, and perpendicular thereto; also making

ν = resultant of the components in direction of the radius,
τ = " " " " of tangent,

and regarding the components which act towards the centre as positive and the contrary negative; also the tangential components which act in the direction $A\,G\,C\,B\,A$ as positive and the contrary negative, we have

$$\nu = \frac{k\,m}{d^2}\cos\varphi - \frac{k\,m}{d^2 + \rho^2 - 2\,d\,\rho\cos\varphi}\,.\cos(\varphi + \theta) \quad . \qquad (78)$$

$$\tau = \frac{k\,m}{d^2}\sin\varphi - \frac{k\,m}{d^2 + \rho^2 - 2\,d\,\rho\cos\varphi}\,.\sin(\varphi + \theta) \quad . \qquad (79)$$

Developing the last factor in equation (78), making $\cos\theta = 1$, because of the small value of θ, we have, after reducing,

$$\nu = -\frac{2\,k\,m\,.\,\rho}{d^3\left(1+\frac{\rho^2}{d^2}-2\frac{\rho}{d}\cos\phi\right)}\,.\,(\cos^2\phi - \frac{\rho}{2\,d}\,.\cos\phi) + \frac{k\,m}{d^2\left(1+\frac{\rho^2}{d^2}-2\frac{\rho}{d}\cos\phi\right)}\,.\sin\phi\,.\sin\theta$$

but from the triangle $E\,G\,S$, we have

$$\sin\theta = \frac{\rho\,.\sin\,(\varphi+\theta)}{d}$$

or neglecting θ in the second member

$$\sin\theta = \frac{\rho\,.\sin\varphi}{d}.$$

Substituting this and omitting all the terms into which $\frac{\rho}{d}$ enters as a factor, which we may do without materially altering the value of ν, we find

$$\nu = -\frac{2\,k\,m\,.\,\rho}{d^3}\,.\cos^2\varphi + \frac{k\,m\,\rho}{d^3}\,.\sin^2\varphi \quad . \quad . \quad . \quad . \quad (80)$$

Again, omitting θ, in the last factor of equation (79), reducing to a common denominator and neglecting the terms of which $\frac{\rho}{d}$ is a factor, we have, after replacing $\cos\varphi\,.\sin\varphi$ by $\frac{1}{2}\sin 2\,\varphi$,

$$\tau = -\frac{k\,m\,\rho}{d^3}\,.\sin 2\,\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

§ 287. Making $\varphi = 0°$ and $\varphi = 180°$ in equation (80), we have the effect on the waters at A and at B; and in both cases

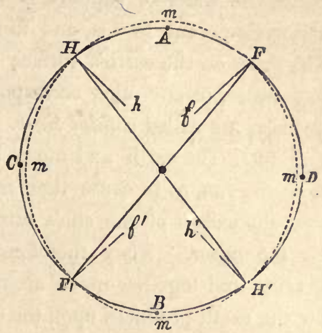
$$\nu = -\frac{2\,k\,m\,.\,\rho}{d^3}.$$

Again, making $\varphi = 90°$ and $\varphi = 270°$, we have the effect on the waters at C and D; and in both cases

$$\nu = \frac{k\,m\,\rho}{d^3}.$$

The values of ν at A and B being negative and those at C and D positive, and these being connected by a law of continuity, through equation (80), the effect of the sun's attraction is to increase the weight of the unit of mass, or, what is the same thing, the specific gravities of all bodies gradually, in both directions, from A to C and D, and to diminish them

Fig. 59.



in like manner from C and D to B. And this being true of all sections of the earth through its centre and the sun, the waters of the ocean on and near the circumference of a section through the earth's centre, and perpendicular to these, will, by the principles of hydraulics, press up those about A and B till their increased height shall compensate for their diminished specific gravity, or till the weights of the balancing columns become equal; so that the ocean surface will tend to assume, as its form of equilibrium, that of an oblongated ellipsoid, of which the longer axis is directed towards the sun. The difference of the longer and shorter semi-axes of this ellipsoid is about 23 inches.

§ 288. If the earth had no diurnal rotation about its axis, this ellipsoid of equilibrium would be formed, and all would be permanent. But the earth's diurnal and orbital motion, together with the inertia of water, leave no sufficient time for this spheroid to be fully formed. Before the waters can take their level, these motions carry the line connecting the earth and sun westwardly, and the place of the vertex of the spheroid of equilibrium in the same direction, thus leaving that of the actual spheroid to the east of the sun, and forcing the ocean to be ever seeking a new bearing. The effect is to produce an immensely broad and excessively flat wave, which follows or endeavors to follow the apparent diurnal motion of the sun, and completes an entire circuit of the earth once in twenty-four solar hours, thus producing a rise and fall of the ocean level twice within this period on every meridian.

§ 289. The rising water is called the *flood*, the falling the *ebb tides*, and the general swell of the ocean is called the *primitive tide-wave.*

§ 290. In the open ocean, where the water is deep, and therefore permits the free transmission of pressure from one remote point to another, the motion is one of oscillation in a vertical direction principally. But where the tide-wave approaches shoals, such as those along the coasts and the beds of estuaries, which intercept the free transmission of pressure, the water becomes piled up, as it were, on the side of the open ocean, without being able to press up any thing to its support on the land side. It therefore flows inland, and produces what are called *derivative flood tides.* After the apex of the tide-wave has passed onward, and low-water suc-

ceeds, the want of support is transferred to the side of the ocean, the water flows out to sea, and forms what are called *derivative ebb tides.* The lines on the earth's surface connecting those places at which high or low water, or any other corresponding phases of the tides, occur simultaneously, are called *cotidal lines.*

§ 291. The earth and moon are so near to each other, and so remote from the sun, as to cause their mutual attractions greatly to predominate over the excess of the sun's attraction for one of them over his attraction for the other. They therefore revolve about their common centre of gravity, and together move around the sun. The attraction of the moon for the earth produces upon the ocean effects similar to those of the sun.

§ 292. The diminution of weight at A and B and increase at C and D vary directly as the attracting masses, and inversely as the cubes of their distance, equations (80) and (81), and the effects upon the tide-wave must be in the same proportion. The mass of the sun is 355000×88 that of the moon, and he is situated at 400 times the moon's distance. Whence the effect of the moon at A and B being

$$\frac{2\,k\,\rho\,m}{d^3},$$

d here distance of a from ♁

that of the sun will be

$$\frac{2\,k\,\rho\,m\;355000 \times 88}{(400)^3\,d^3};$$

and dividing the last by the first, we have

$$\frac{355000 \times 88}{(400)^3} = 0.488;$$

so that the effect of the moon is more than double that of the sun.

§ 293. The lunar day exceeds the solar on an average about 50 minutes; the lunar tide must therefore move slower than the solar by about $12^\circ.5$ in 24 solar hours; and hence they must sometimes conspire and sometimes oppose one another. The former occurs when the angular distance of the sun from the moon, as seen from the earth, is 0° or 180°, and the latter when this distance is 90°.

This alternate reinforcement and partial destruction of the lunar by the solar wave, produce what are called *spring* and *neap* tides; the former being their sum, the latter their difference.

§ 294. The sun and moon, by virtue of the ellipticities of the terrestrial and lunar orbits, are alternately nearer to and further from the earth than their mean distances.

If the mean distances of the sun and moon be substituted in Eq. (80), the corresponding ellipticities of the solar and lunar spheroids will be found to be 2 and 5 feet respectively; so that the average spring tide will be to the average neap, as $5 + 2$ to $5 - 2$, or as 7 to 3.

Substituting the greatest and least distance of the sun in the same equation, the resulting tides are called respectively *apogean* and *perigean* tides; and representing the ellipticity of the solar spheroid at the mean distance by 20, the corresponding ellipticities become 19 and 21. In like manner the ellipticities of the lunar spheroid will be found to vary between the limits 43 and 59. Hence, the highest spring tide will be to the lowest neap, as $59 + 21$ is to $43 - 21$, or as 10 to 2,8.

§ 295. The sun and moon act to form the apexes of their respective tide-waves at different places, depending upon their angular distances apart. This gives rise to a resultant wave, whose apex is at some intermediate place, and the actual *tide day*, or interval between the occurrences of two consecutive maxima of the resultant wave at the same place, will vary as the component waves approach to or recede from one another. This variation from uniformity in the length of the tide day is called the *priming* or *lagging of the tides*—the former indicating an acceleration and the latter a retardation of the recurrence of high-water at the same place. The priming and lagging are particularly noticeable about the time the angular distance between the moon and sun is 0° or 180°, that is, as we shall presently see, about new or full moon.

§ 296. The effort of the attracting body being to form the nearest vertex of its aqueous spheroid immediately under it, the summit of the lunar and solar tide-waves follow the course of the moon and sun to the north and south of the equator, and this gives rise to a monthly and annual variation in the heights of the principal tides at a given place.

§ 297. But of all causes of difference in the heights of tides, local situation is the most influential. In some places, the tide-wave rushing up narrow channels becomes so compressed laterally as to be elevated to extraordinary heights. At Annapolis, in the Bay of Fundy, it is said to rise 120 feet.

§ 298. Were the waters of the ocean free from obstructions due to viscosity, friction, narrowness of channels leading to different ports, and the like, the time of high-water at a given place, would depend only upon the relative positions of the sun and moon, and their meridian passages. But all these causes tend to vary this time, and to postpone it unequally at different ports. This deviation of the time of actual from that of theoretical high-water at any place, is called the *establishment of the port*, and is

an element of the highest maritime importance. When ascertained from observation, it enables the mariner to know by simply noticing the places of the sun and moon with reference to the meridian, when he may safely attempt the entrance of a port obstructed by shoals.

§ 299. In bays, rivers, and sounds, where tides arise from an actual flow of water, the time of "*Slack water*," or stagnation, must not be confounded with that of high and low water. They may, indeed, coincide, but not of course. A river current, for instance, and another from the sea, may neutralize each other's flow, while both conspire to elevate the water surface; so, also, an ebbing current may continue its onward course after the more advanced part of a returning flood has put its surface on the rise by checking its velocity. The same of two currents meeting in a sound.

§ 300. Starting from A as an origin (Fig. 58), and proceeding in the direction of $A\ C\ B\ D\ A$, we find the value of τ, Eq. (81), negative in the 1st and 3d quadrants, and positive in the 2d and 4th; so that the tangential components of the solar and lunar attractions conspire with the normal to increase the height of the great tide-waves by impressing upon the water a motion of translation towards their apexes. But before the inertia of the water will permit the latter to acquire much velocity, the rotary motion of the earth reverses the direction of the impelling forces, and the final effect due to this cause is, in consequence, but small.

TWILIGHT.

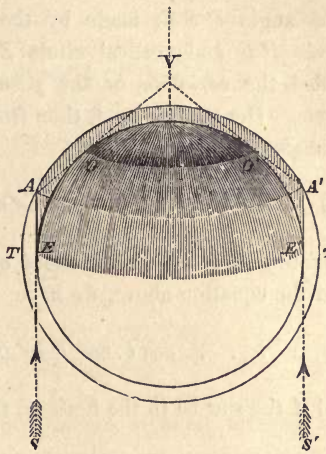
§ 301. The curve along which a conical surface, tangent to the sun and earth, is in contact with the latter body, is called the *circle of illumination*. It divides the dark from the enlightened portion of the earth's surface, and is ever shifting its place by the diurnal motion.

§ 302. The base of the earth's shadow, into which a spectator enters at sunset, and from which he emerges at sunrise, is inclosed by an atmospheric wall-like ring, illuminated by the direct light from the sun, immediately exterior to that which just grazes the earth's surface. The light is reflected from the particles of this ring into the shadow, and gives to the air about its boundary a secondary and partial illumination called *Twilight*. A conical surface through the summit of this ring, and tangent to the earth, determines, by its contact with the latter, a limit within which the twilight cannot sensibly enter, and twilight will only continue while the spectator is carried by the earth's diurnal motion across the zone of which this line is the inner, and the circle of illumination the exterior boundary. The

belt of the earth's surface over which twilight is visible, is called the *crepuscular zone.*

Thus, let $E\,O\,O'\,E'$ be a section of the earth's surface on the opposite side from the sun; $T\,A\,A'\,T'$ of the atmosphere by the same plane, the height of the air being exaggerated to avoid confusing the figure; and $S\,A$ and $S'\,A'$ two solar rays tangent to the earth's surface. The particles of air in $E\,A\,T$ and $E'A'T'$ will be illuminated, while those in the space $E\,A\,A'\,E'$ will be in the shadow. The section will cut from the tangent cone the elements $A\,V$ and $A'\,V$, which touch the earth at O and O', respectively, and being revolved about the line connecting the centres of the earth and sun, the part $E\,A\,T$ will generate the luminous atmospheric inclosure and the points E and O, the circle of illumination and interior boundary of the crepuscular zone, respectively.

Fig. 60.



§ 303. To a spectator within the crepuscular zone a portion only of the illuminating ring will be visible, and will appear as a bright elliptical segment, with its chord in the horizon, its vertex in the vertical circle through the sun, and its outline almost lost in the gradual decay of light produced by the diffusive action of the air and the progressive thinning and consequent diminution in the number of reflecting particles towards the summit of the luminous ring.

§ 304. When the spectator is carried obliquely through the crepuscular zone without crossing its smaller base, twilight will last all night.

§ 305. Resuming Eq. (74), that is

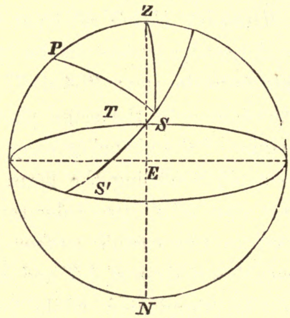
$$\cos z = \sin l \sin d + \cos l \cos d \cos P;$$

substituting the latitude of the place for l, the declination of the sun for d, and the value of P, obtained by converting the observed time from noon to the end of twilight in the evening, or from the beginning of twilight in the morning till noon, into degrees, the average value of a number of determinations for z will be found to be about $108°$; so that at the end of evening or beginning of morning twilight the sun is $18°$ below the horizon.

§ 306. From the above equation we find

$$\sin l = \frac{\cos z - \cos l \,.\, \cos d \,.\, \cos P}{\sin d}.$$

Fig. 61.



The angle $P\,S\,Z$, made by the hour circle $P\,S$ and vertical circle $Z\,S$, is called the *variation* or the *parallactic angle.* Denote this by ξ, then from the triangle $Z\,P\,S$, will

(1) . . . $\sin l = \sin d \cos z + \cos d \sin z \cos \xi.$

Equating the second members of this and the equation above, we have

(2) . . . . $\cos l \,.\, \cos P = \cos z \,.\, \cos d - \sin z \sin d \,.\, \cos \xi$;

and if the sun be in the horizon, then will

$$z = 90°,\ P = P',\ \text{and}\ \xi = \xi',\ \text{and}$$

(3) . . . . $\cos l \,.\, \cos P' = - \sin d \,.\, \cos \xi'$.

Also, from the same triangle,

(4) . . . . $\cos l \,.\, \sin P = \sin z \,.\, \sin \xi$;

and when the sun is in the horizon,

(5) . . . . $\cos l \,.\, \sin P' = \sin \xi'$.

Multiply (2) by (3), also (4) by (5), and add the products, there will result,

$$\cos^2 l \,.\, \cos (P - P') = - \cos z \cos d \sin d \cos \xi' + \sin z \cos (\xi - \xi') - \cos^2 d \sin z \cos \xi \cos \xi'.$$

From (1), we have

$$\cos \xi = \frac{\sin l - \sin d \,.\, \cos z}{\cos d \,.\, \sin z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

and for the sun in the horizon

$$\cos \xi' = \frac{\sin l}{\cos d}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

which substituted above, give

$$\cos^2 l \, . \cos (P - P') = \sin z \, . \cos (\xi - \xi') - \sin^2 l;$$

whence, because

$$\cos (P - P') = 1 - 2 \sin^2 \tfrac{1}{2} (P - P'),$$

we have

$$\sin^2 \tfrac{1}{2} (P - P') = \frac{1 - \sin z \, . \cos (\xi - \xi')}{2 \cos^2 l};*$$

passing to the arc and making

$$t = \frac{P - P'}{15},$$

we have

$$t = \frac{2}{15} \sin^{-1} \sqrt{\frac{1 - \sin z \, . \cos (\xi - \xi')}{2 \cos^2 l}} \quad . \quad . \quad (84)$$

which will give the time required for the sun, or other heavenly body, to pass from the horizon to a zenith distance z, or, conversely, from a zenith distance z to the horizon.

Making $z = 90° + 18° = 108°$, Eq. (84) becomes

$$t = \frac{2}{15} \, . \sin^{-1} \sqrt{\frac{1 - \cos 18° \, . \cos (\xi - \xi')}{2 \cos^2 l}} \quad . \quad . \quad . \quad (85)$$

which will give the duration of twilight for any latitude and season of the year; and for this purpose, the values of ξ and ξ' must be found from Eqs. (82) and (83), after making, in the former, $z = 90° + 18°$.

The value of t, in Eq. (85), becomes a minimum when $\xi = \xi'$, and for the duration of the shortest twilight, we have, after replacing $1 - \cos 18°$ by its equal $2 \sin^2 9°$,

$$t = \frac{2}{15} \, . \sin^{-1} (\sin 9° \, . \sec l) \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

Equating the second members of Eqs. (82) and (83)

$$\sin d = - \tan 9° \, . \sin l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

In a given latitude, Eq. (86) will make known the shortest twilight, and Eq. (87) the season at which it will occur.

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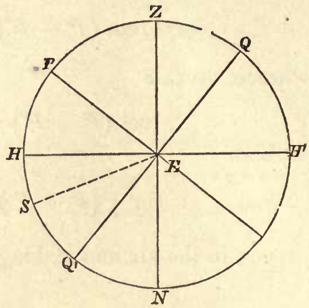
* Ann Arbor Astronomical Notices, N . 1.

Fig. 62.



§ 308. The sign of the second member of Eq. (87) shows that at the time of shortest twilight the spectator and the sun will be on opposite sides of the plane of the equinoctial.

§ 309. The depression of the lowest point Q' of the equinoctial below the horizon HH', is $90° - l$; and of the lowest point S of the sun's diurnal path, when his declination is of the same name as the spectator's latitude, $90° - (l + d)$; and when

$$90° - (l + d) = 18°,$$

the end of the evening will be the beginning of morning twilight, and the nocturnal path of the spectator will be tangent to the inner boundary of the crepuscular zone.

THE SUN.

§ 310. The *Sun*, as before stated, is the central body of the solar system, and from this circumstance gives to the latter its name. It occupies one of the foci of all the elliptical orbits of the planets, and, of course, that of the earth.

§ 311. *Distance and Dimensions of the Sun.*—Its horizontal parallax denoted by P, and apparent semi-diameter denoted by s, vary inversely as the earth's radius vector. For the mean radius it is found, § 113–6,

$$P = 8''.6, \text{ and } s = 16'\ 01''.5;$$

which in Eqs. (28) and (29) give

$$r_n = \rho \cdot \frac{\omega}{P} = \rho \cdot \frac{206264''.8}{8''.6} = 23984 \cdot \rho \quad . \quad . \quad . \quad (88)$$

$$d = \rho \cdot \frac{16'\ 01''.5}{8''.6} = \rho \cdot \frac{961''.5}{8''.6} = 111.5\ \rho \quad . \quad . \quad . \quad . \quad (89)$$

From Eq. (88) it appears that the mean distance of the earth from the sun is 23984 times the earth's equatorial radius; and from Eq. (89) that the sun's diameter is 111.5 times that of the earth. The volumes of these bodies are as the cubes of their diameters, and hence the volume of the sun is 1384472 times that of the earth.

§ 312. If the equatorial radius ρ be replaced in Eqs. (88) and (89) by its value in miles, § 98, we find

$$r_n = 95{,}043{,}800 \text{ miles,}$$
$$2d = \quad 882{,}000 \quad \text{"} \quad ;$$

that is to say, the mean distance of the earth from the sun is, in round numbers, about 95 millions of miles, and the diameter of the sun is 882 thousand miles. The mean distance of the earth from the sun is assumed as the unit of linear dimensions in all celestial measurements.

§ 313. *Mass of Sun.*—In Analytical Mechanics, § 201, we find the equation

$$T = \frac{2\pi \cdot a^{\frac{3}{2}}}{\sqrt{k}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)'$$

in which T denotes the periodic time of a body revolving about a centre of attraction, a the mean distance of the body from the centre, π the ratio of the circumference to the diameter, and k the attraction on a unit of mass at the unit's distance.

Let k become μ in the case of the sun's action on the earth; then will T become the sidereal year, and a the semi-transverse axis of the earth's orbit, and

$$\mu = \frac{4\pi^2 \cdot a^3}{T^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

and for the action of the earth upon the moon

$$\mu' = \frac{4\pi^2 \cdot a'^3}{T'^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

in which μ' denotes the attraction on the unit of mass at the unit's distance exerted by the earth.

Now the attractions exerted by two bodies on the same mass at the same distance, are directly proportional to their masses respectively; and denoting the mass of the sun by M, and that of the earth by M' we have

$$\frac{M}{M'} = \frac{\mu}{\mu'} = \frac{T'^2}{T^2} \cdot \frac{a^3}{a'^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

But in Eqs. (62) and (88)

$$T = 365^d.25, \text{ and } a = 23984 \cdot \rho;$$

and we shall presently see that the moon revolves about the earth once in 27.5 days, at a mean distance of 60 times the equatorial radius of the earth. Making, therefore,

$$T' = 27.5, \text{ and } a' = 60 \,.\, \rho,$$

and substituting above, we have

$$\frac{M}{M'} = 354936.$$

That is, the sun contains 354,936 times as much matter as the earth; and as the common centre of inertia divides the line joining their respective centres of inertia into two parts, which are inversely proportional to their masses, the common centre of inertia of the sun and earth, about which both bodies would describe their respective orbits were they undisturbed by the other bodies of the system, is but 267 miles from the sun's centre, or about $\frac{1}{3300}$th part of its own diameter.

§ 314. Denote by D the density of the sun, and by V its volume; also by D' and V', respectively, the density and volume of the earth; then Analytical Mechanics, § 18,

$$M = D \,.\, V,$$

$$M' = D' \, V';$$

and by division

$$\frac{M}{M'} = \frac{D \,.\, V}{D' \, V'};$$

and substituting the ratio of the masses and of the volumes given above, we find

$$D = 0.2543 \,.\, D';$$

so that the sun is but a trifle more than one-quarter as dense as the earth. The latter is known, from the recent experiments of Mr. Francis Baily, to be 5.67, the density of water being unity; and this value substituted for D' above, makes the density of the sun not quite once and a half that of water.

§ 315. *Surface Gravitation of the Sun.*—By the laws of gravitation, the attraction of one body upon another varies as the quantity of matter in the attracting body directly, and the square of the distance through which the attraction is exerted, inversely. The distance is that between the centres of gravity of the bodies.

Denote by W and W' the weights of the same body on the surfaces of the sun and earth, respectively; then will

$$W : W' :: \frac{M}{d^2} : \frac{M'}{\rho^2};$$

whence

$$\frac{W}{W'} = \frac{M}{M'} \cdot \frac{\rho^2}{d^2}; \quad \ldots\ldots \quad (93)$$

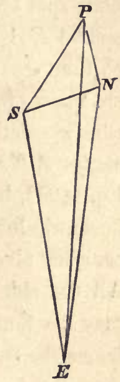
and substituting the values just found,

$$\frac{W}{W'} = 28{,}5.$$

That is, a body weighing one pound at the equator of the earth would weigh 28,5. pounds at that of the sun; and acquire, therefore, during each second of its fall a velocity of 916,44 feet.

§ 316. *Sun's Rotation and Axis.*—Through the telescope the sun's surface often exhibits dark spots which slowly change their places and figure. They cross the solar disk from east to west, and thus reveal a rotary motion of the sun itself from west to east about an axis.

Fig. 63.



§ 317. To find the time of rotation and the position of the axis, it will be necessary first to find the heliocentric longitudes and latitudes of the same spot at different times. To do this, let S be the sun's centre, E that of the earth, P the spot, and N its projection upon the plane of the ecliptic. Make

$l =$ heliocentric longitude of the earth;
$x =$ " " " spot;
$y = PSN =$ heliocentric latitude of spot;
$\beta = PEN =$ geocentric latitude of spot;
$e = SEN =$ difference of geocentric longitude of the sun and the spot,
$\Delta =$ sun's apparent semi-diameter.

Then $$SP \sin y = PN = EP \sin \beta = SE \sin \beta,$$

because the difference between EP and SE is insignificant in comparison with either; whence

$$\sin y = \frac{SE}{SP} \cdot \sin \beta = \frac{\sin \beta}{\sin \Delta} \quad . \quad . \quad . \quad . \qquad (94)$$

Again

$$SP \cdot \cos y : EP \cdot \cos \beta :: SN : NE,$$
$$:: \sin e : \sin (l - x);$$

whence

$$\sin (l - x) = \frac{\sin e \cdot \cos \beta}{\cos y} \cdot \frac{EP}{SP},$$

$$= \frac{\sin e \;\; . \cos \beta}{\sin \Delta \cdot \cos y};$$

and replacing $\cos y$ by its value,

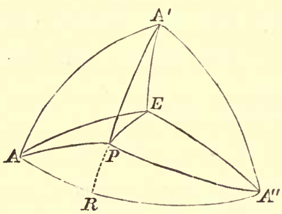
$$\sin (l - x) = \frac{\sin e \cdot \cos \beta}{\sqrt{\sin^2 \Delta - \sin^2 \beta}};$$

or for logarithmic computation,

$$\sin(l - x) = \frac{\sin e \cdot \cos \beta}{\sqrt{\sin(\Delta + \beta) \cdot \sin(\Delta - \beta)}} \quad . \; . \quad (95)$$

§ 318. *Position of the Sun's equator, and the time of the Sun's rotation.* Let E be the pole of the ecliptic, P that of the sun's equator; A, A', and A'' the heliocentric places of the same spot observed at three different times; and let EA, EA', EA'', PA, PA', PA'' be the arcs of great circles. The first three are known from Eq. (94), being the heliocentric colatitudes of the spot; as also the angles AEA', AEA'', and $A'EA''$ from Eq. (95), being the differences of the heliocentric longitudes—all deduced from geosurface observations of the spot's right ascension and declination, § 152.

Fig. 64.



All the sides and angles of the triangles AEA', AEA'', and $A'EA''$ may be found, two sides and the included angle in each being given; hence the sides AA', $A'A''$, and $A''A$, and the angles A, A', and A'', in the triangle $AA'A''$, are known. Now P being the pole of the sun's equator, parallel to which the spot revolves,

$$PA = PA' = PA'';$$

Make

$$2S = A + A' + A'' = 2PAR + 2PA'A + 2PA'A''$$
$$= 2PAR + 2A';$$

whence

$$PAR = S - A',$$

and PAR becomes known.

If PR be perpendicular to AA'',

$$AR = \tfrac{1}{2} AA'';$$

then in the right-angled triangle APR, the angle at A and the side AR being known, the side PA is computed; and, finally, in the triangle APE, the sides AP and AE, and the angle $EAP = EAA'' - PAA''$ being known, PE is computed.

§ 319. The arc EP is the heliocentric colatitude of the pole of the sun's equator, and the angle AEP, added to the heliocentric longitude of the spot at A, gives its heliocentric longitude. The position of the sun's equator becomes, therefore, known. The heliocentric latitude and longitude of its north pole at the beginning of the present century were, respectively, 82° 30′ and 350° 21′.

From the triangle APR the angle APR becomes known, the double of which is APA''. Then, denoting by T the time of one rotation, and by t the interval between the observations on the spot at A and A'', we have

$$APA'' : t :: 360° : T;$$

whence T is known to be about 25.325 days, making the angular velocity of the sun around its axis about one twenty-fifth that of the earth.

From this motion it is concluded that the sun is flattened at its poles.

§ 320. *Physical constitution of Sun.*—The study of the solar spots has led to interesting conclusions in regard to the physical constitution of the sun itself. The spots are transient in character, variable in size, shape and number, and confined to two comparatively narrow zones parallel to, and at no great distance from the sun's equator. They appear perfectly black, and surrounded by a border less dark, called a *penumbra.* The black part and penumbra are distinctly defined in outline, and do not fade the one into the other. Sometimes this penumbra presents two or more shades, and in this case also there is no gradation, but well-marked outline, indicating a total absence of blending.

Fig. 65.

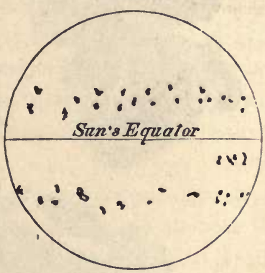
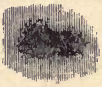
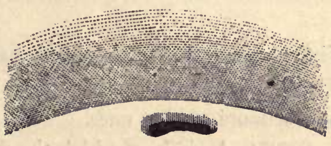


Fig. 66.



As the spots move towards the edge of the sun, the penumbra on the inner side gradually contracts, and with the black spot disappears before reaching the boundary of the disk; the penumbra on the outer side expands, and is the last visible remnant of the spot as it passes behind the sun. At its reappearance on the opposite edge of the sun, the spot exhibits similar phenomena—the penumbra first appears, then the black portion on its inner side, the contraction of the penumbra in width, and its extension around the black till the latter is entirely surrounded.

Fig. 67.

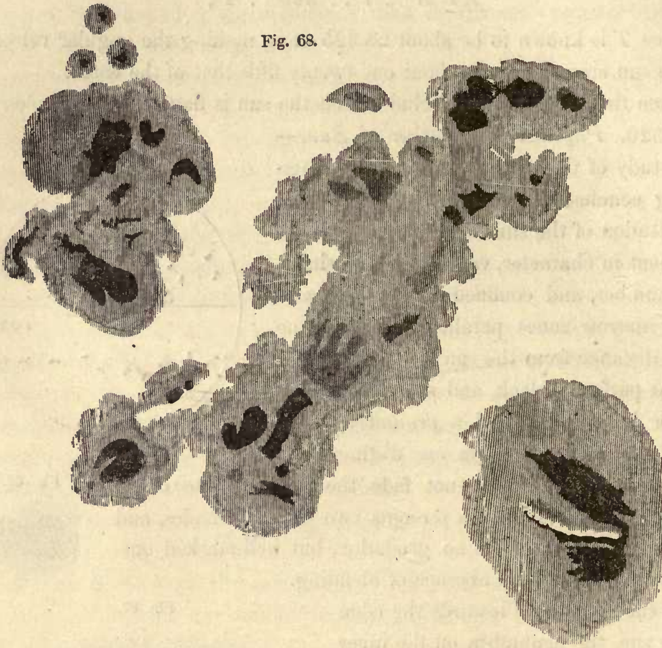


This is precisely the appearance that would be presented by a deep pit or excavation with a dark or non-luminous bottom. The rotation of the sun would bring the slanting surface leading from the inner edge of its mouth more and more in the direction of the spectator till it would be lost in the foreshortening, the inner edge would presently mask the bottom, and the surface of the opposite side would be turned so nearly perpendicu-

larly to the line of sight as to appear broadest just before passing behind, at disappearance, or at reappearance, to the front of the sun.

§ 321. The spots gradually expand or contract, change their figure, vanish, and break out again at new places where none were before. When

Fig. 68.



disappearing, the central black part contracts to a point and vanishes before the penumbra; and a single spot is sometimes seen to break up into two or more smaller ones.

§ 322. A circle of which the diameter is one second is the smallest visible area. A single second at the earth is subtended at the sun by a distance of 461 miles, and the area of the least visible circle on the sun's surface is, therefore, 167,000 square miles. A spot whose diameter was 45,000 miles has been known to close up and disappear in course of six weeks, thus causing the edges to approach one another at the rate of 1000 miles a day. Many spots distinctly visible have been observed to vanish in a few hours, indicating a degree of mobility inconsistent with the idea of solids and liquids.

§ 323. Light proceeding very obliquely from the surfaces of incandescent solids and liquids is always polarized, whereas that from gases under the same circumstances is not. The light from the edge of the solar disk

leaves the surface of the sun in a direction nearly coincident with the surface itself, and yet when examined by the usual tests exhibits no signs of polarization.

§ 324. The luminous part of the sun is not uniformly bright, but presents a mottled appearance, and immediately about the spots are often seen well-defined and branching streaks, called *facules*, brighter than other parts of the surface; among these, spots often make their appearance. They are best seen near the border of the disk.

§ 325. The brightness of the solar disk sensibly diminishes towards the borders; and this fact has given rise to the supposition that the sun is surrounded by an atmosphere not perfectly transparent, and of great extent above the luminous envelope. The loss of light towards the borders would result from the greater absorption of the luminiferous waves in consequence of traversing a greater thickness of the atmosphere in that direction.

§ 326. The moon, of which an account will be given presently, is known to be a non-luminous, opaque, spherical mass, and so near the earth as to give to it an apparent diameter about equal to that of the sun. This little body often interposes itself so as completely to conceal the sun from view, producing what is called a solar eclipse. At the instant of greatest solar obscuration—that is, when the moon completely covers the sun—red protuberances resembling flames of fire are seen to issue apparently from the edge of the moon, but in fact from that of the sun, revealing the existence of intense commotion and physical changes about the surface of the latter body.

§ 327. From all which it is inferred that the sun is an opaque solid, covered by a gaseous envelope of well-defined boundary and intense luminosity, the whole being surrounded by a non-luminous atmosphere of vast extent.

No explanation free from objection has, thus far, been given for the solar spots. Some have supposed them to arise from scoria or flakes of incombustible matter floating upon the sun's surface; while others, with perhaps greater reason, have attributed them to temporary openings in the photosphere that envelops the sun, exposing to view detached portions of his solid crust, which appear black from contrast.

But it must not be inferred from this that the solid portion of the sun is regarded as non-luminous. Were he stripped of his gaseous coating, he would no doubt shine with diminished but yet intense brilliancy. A piece of quicklime, in a state of most active combustion under the action of a compound blowpipe, is, when projected upon the bright part of the sun, as dark as the darkest part of the spots.

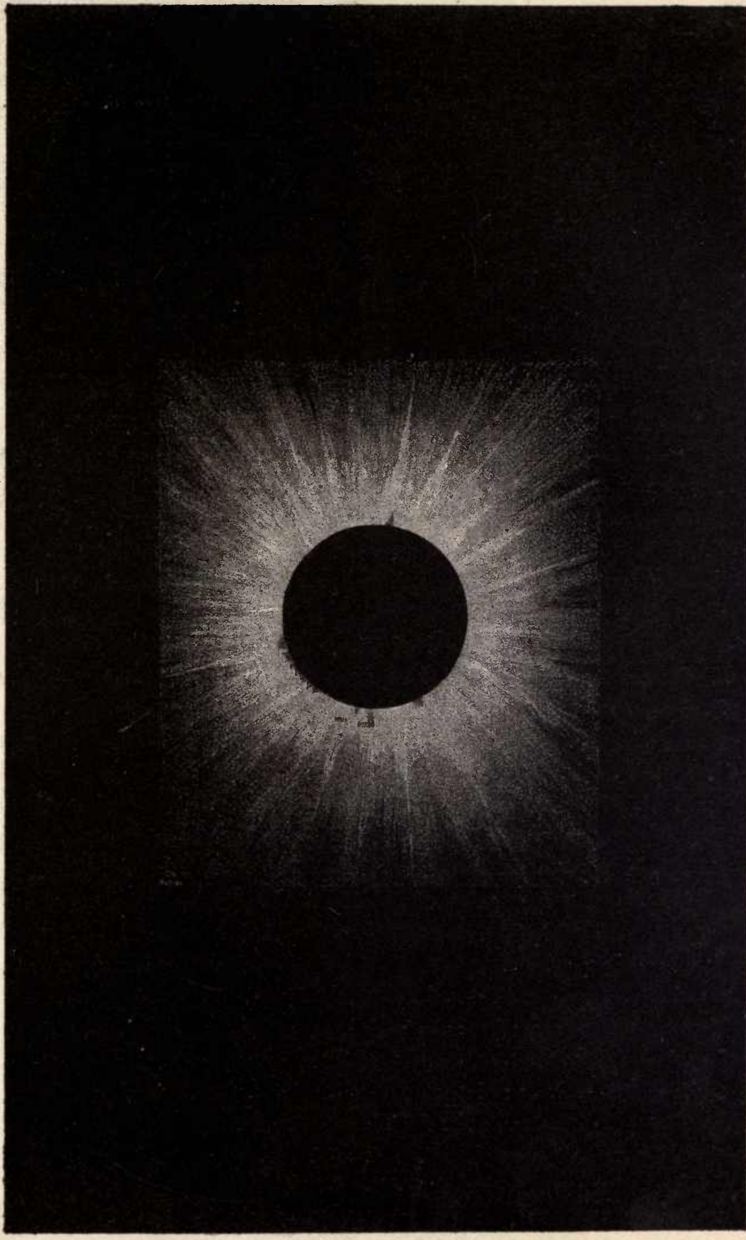
During the interposition of the lunar screen between the sun and a

spectator on the earth, the surrounding landscape takes on the obscure illumination produced by a closing evening twilight, and the temperature is always sensibly depressed, thus corroborating the suggestions of other phenomena, that the sun is the great source of light and heat to the earth.

But light and heat are the results of molecular agitation. What, then, is the cause of that *perpetual* molecular vibration essential to the self-luminosity of the sun? The solar system is believed to have resulted from the subsidence of a vast nebula; the planets and satellites are detached fragments left behind in the progress of the general mass towards the centre; the sun itself is the central accumulation. This nebula must have extended originally far beyond the orbit of Neptune, the exterior planet now known. The distance of this planet from the sun is more than thirty times that of the earth. The condensation has taken place under the action of weight impressed upon the elements by their reciprocal attractions for one another. The living force with which so much matter would reach the terminus of a fall necessary to transfer it to its present abode, could not fail to impress upon the condensed mass the most intense molecular agitation. This agitation, or molecular living force, can only be lost through the agency of the surrounding medium which diffuses it through space; and the loss in a given time is determined by the density of the medium, being less as the density is less. The medium which pervades the planetary space is so attenuated as to offer no sensible resistance to the denser bodies that move through it, nor could we be conscious of its existence at all but for the almost inconceivably small amount of living force which it brings from the sun to impress upon us the sensations of light and heat. A process so slow would require countless ages to bring the solar molecules to rest, and convert the sun into a non-luminous mass.

PLANETS.

§ 328. Let us now resume the Planets. As before remarked, these bodies move in elliptical curves, of which one of the foci of each is at the centre of the sun. A spectator on the earth views these bodies, therefore, from a station far removed from their centre of motion, and even from the planes of their orbits. Hence, their co-ordinates of place, measured by the aid of instruments, are affected with both geocentric and heliocentric parallaxes. To eliminate these, and then from the resulting heliocentric co-ordinates to determine the elements of a conic section whose curve shall pass through the observed places and have a focus at the sun's centre, is the object of one of the most important problems in Astronomy.



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Three observed right ascensions and declinations, together with the intervals of time between the observations, are sufficient for its solution.

§ 329. The planes of the orbits passing through the sun, the orbits themselves will pierce the plane of the ecliptic in two points, called *nodes*. The node by which the body passes from the south to the north of the ecliptic is called the *ascending node;* the other is called the *descending node.*

§ 330. The angle which the plane of a body's orbit makes with that of the ecliptic or equinoctial, is called the *inclination.*

§ 331. The semi-transverse axis, called the *mean distance*, and eccentricity, determine the size and shape of the conic section.

§ 332. The inclination, heliocentric longitude, or right ascension of the ascending node, and of the perihelion, fix the position of the orbit in space.

§ 333. The *time* of the body's being at perihelion, and its mean angular velocity, called its *mean motion*, give the circumstances of the body's motion in the orbit.

§ 334. The orbit of a heavenly body is therefore completely determined when the *inclination*, *mean distance*, *eccentricity*, *longitude of the ascending node*, *longitude of the perihelion*, *epoch of the perihelion passage*, and *mean motion* are known. These are called the elements of an orbit. They are seven in number.

§ 335. *To find a planet's elements.*—The polar equation of the orbit is

$$r = \frac{a(1 - e^2)}{1 + e\cos v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

in which r is the radius vector of the planet, a the semi-transverse axis, called the planet's *mean distance*, e the eccentricity, and v the planet's angular distance from perihelion, called the *true anomaly;* the pole being at the sun.

Making $v = 90°$, r becomes the *semi-parameter*, which denote by L, and we have, Eq. (96), and *Analyt. Mechanics*, § 200,

$$L = a(1 - e^2) = \frac{4c^2}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

in which c denotes the area described by a radius vector in a unit of time; and Eq. (96) may be written

$$r = \frac{L}{1 + e\cos v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

whence

$$e\cos v = \frac{L}{r} - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

from which, denoting the planet's velocity in the direction of the radius vector by V_r, we find, Appendix VI.,

$$e \sin v = \frac{L}{2c} \cdot V_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

which divided by Eq. (99) gives

$$\tan v = \frac{L}{2c} \cdot \frac{r}{L - r} \cdot V_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

§ 336. Denote by p, the perihelion distance, then, making $v = 0$, in Eq. (96),

$$p = a(1 - e) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

§ 337. Denoting by T the periodic time, we have, *An. Mec.* § 201,

$$T = \frac{2\pi \cdot a^{\frac{3}{2}}}{\sqrt{k}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

and denoting the *mean motion* by n,

$$n = \frac{2\pi}{T} = \frac{\sqrt{k}}{a^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

§ 338. Take an auxiliary angle, such that

$$\cos v = \frac{\cos u - e}{1 - e \cos u} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

then, Appendix VII., $\quad n\,t = u - e \sin u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$

in which t denotes the time from perihelion, and n, as above, the *mean motion*.

§ 339. The product $n\,t$ is the angular distance which the planet would be from perihelion had it moved from that point with its mean motion n, and is called the *mean anomaly*.

§ 340. The auxiliary angle u is called the *eccentric anomaly*, and differs from $n\,t$ only because of the eccentricity of the orbit; for if the latter be zero, $n\,t$ will equal u.

§ 341. From Eq. (105) we readily find

$$\tan \frac{u}{2} = \sqrt{\frac{1 - e}{1 + e}} \tan \frac{v}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

§ 342. Making in Eq. (103), $k = \mu$, $a = 1$, and $T = 365^{d}.256$, we find

$$\log \mu = 6.4711640$$

$$\log \sqrt{\mu} = 8.2355820$$

§ 343. From the centre of the sun draw right lines respectively to the vernal equinox, intersection of the solstitial colure with the equinoctial, and north celestial pole, and take these as the axes x, y, and z. The planes of the equinoctial, of the equinoctial colure, and of the solstitial colure, will be the co-ordinate planes $x\,y$, $x\,z$, and $z\,y$ respectively.

Denote by V_x, V_y, and V_z the component velocities of the planet in the direction of the axes, and by c', c'', and c''' the projections of c on the co-ordinate planes $x\,y$, $z\,y$, and $z\,x$ respectively; then, *Analytical Mechanics*, § 189, equations (260), will

$$\left.\begin{aligned} xV_y - yV_x &= 2\,c', \\ yV_z - zV_y &= 2\,c'', \\ zV_x - xV_z &= 2\,c''', \end{aligned}\right\} \quad \ldots\ldots\ (108)$$

and

$$c^2 = c'^2 + c''^2 + c'''^2 \quad \ldots\ldots\ (109)$$

§ 344. Denote the inclination of the orbit to the plane of the equinoctial by i, then will

$$\cos i = \frac{c'}{c} \quad \ldots\ldots\ (110)$$

§ 345. Also,

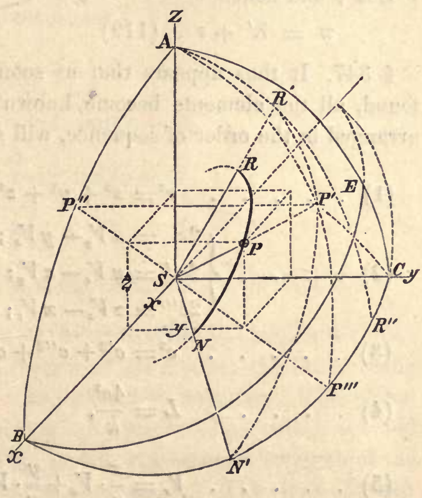
$$r^2 = x^2 + y^2 + z^2 \quad \ldots\ldots\ (111)$$

and, Appendix VIII.,

$$V_r = \frac{x}{r} \cdot V_x + \frac{y}{r} \cdot V_y + \frac{z}{r} \cdot V_z \quad \ldots\ldots\ (112)$$

§ 346. Let S be the sun, P the place of the planet, R that of the perihelion, B the vernal equinox, E the summer solstice, A the north celestial pole, $B\,E$ the ecliptic, $R'P'N'$ the intersection of the plane of the planet's orbit with the celestial sphere, N' the heliocentric place of the ascending node N on the equinoctial, $A\,P'P'''$ and $A\,R'R''$ quadrants of great circles of the celestial sphere. Make

Fig. 70.



$\lambda = B\,P''' =$ the planet's heliocentric right ascension.

$\delta = A\,P' =$ the planet's heliocentric north polar distance.

$\eta = N'P''' =$ distance in heliocentric right ascension from the node.

$\varsigma = B\,N' =$ heliocentric right ascension of ascending node.

$\varphi = N'P' =$ distance of the planet from the node.

$\lambda' = N'R'' =$ distance of perihelion in right ascension from the asc. node.

$\varpi = B\,R'' =$ heliocentric right ascension of the perihelion.

Then $$\tan\lambda = \frac{y}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

$$\tan P''SB = \tan P'CP''' = \cot P'CA = \frac{z}{x};$$

and in the triangle $AP'C$, the side AC being 90°,

$$\cot\delta = \cos\lambda \cdot \frac{z}{x} \quad (114)$$

Again, in the triangle $P'P'''N'$, right-angled at P''',

$$\sin\eta = \cot\delta \cdot \cot i \quad (115)$$

$$\varepsilon = \lambda - \eta \quad (116)$$

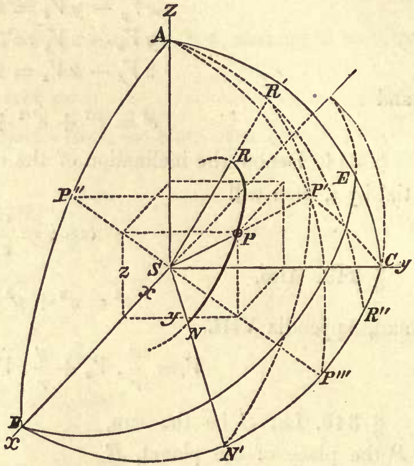
$$\tan\varphi = \sec i \cdot \tan(\lambda - \varepsilon) \quad (117)$$

In the triangle $N'R'R''$, right-angled at R'',

$$\tan\lambda' = \cos i \,.\, \tan(\varphi + v) \quad (118)$$

in which v is the true anomaly $P'SR'$; and hence

$$\varpi = \lambda' + \varepsilon \quad (119)$$



§ 347. It thus appears that as soon as x, y, z, V_x, V_y and V_z are found, all the elements become known; and the preceding formulas, arranged in the order of sequence, will stand

(1) . . . . . $r^2 = x^2 + y^2 + z^2$;

(2) . . . . $\begin{cases} 2c' = xV_y - yV_x; \\ 2c'' = yV_z - zV_y; \\ 2c''' = zV_x - xV_z; \end{cases}$

(3) . . . . . $c^2 = c'^2 + c''^2 + c'''^2$.

(4) . . . . . $L = \dfrac{4c^2}{k}$,

(5) . . . . . $V_r = \dfrac{x}{r} \cdot V_x + \dfrac{y}{r} \cdot V_y + \dfrac{z}{r} \cdot V_z$.

(6) . . . . $\tan v = \dfrac{L}{2c} \cdot \dfrac{r}{L - r} \cdot V_r$.

(7) . . . . $e = \dfrac{1}{\cos v} \cdot \left(\dfrac{L}{r} - 1\right)$; Eq. (99).

(8) . . . . . $a = \frac{r\,(1 + e\cos v)}{1 - e^2}$; Eq (96).

(9) . . . . . $p = a\,(1 - e)$.

(10) . . . . . $n = \frac{\sqrt{k}}{a^{\frac{3}{2}}}$.

(11) . . . . . $\cos u = \frac{\cos v + e}{1 + e\cos v}$; Eq. (105).

(12) . . . . . $t = \frac{u - e\sin u}{n}$; Eq. (106).

(13) . . . . . $\cos i = \frac{c'}{c}$.

(14) . . . . . $\tan\lambda = \frac{y}{x}$.

(15) . . . . . $\cot\delta = \cos\lambda \,.\, \frac{z}{x}$.

(16) . . . . . $\sin\eta = \cot\delta \,.\, \cot i$.

(17) . . . . . $\varepsilon = \lambda - \eta$.

(18) . . . . . $\tan\varphi = \sec i \,.\, \tan(\lambda - \varepsilon)$.

(19) . . . . . $\tan\lambda' = \cos i \,.\, \tan(\varphi + v)$.

(20) . . . . . $\varpi = \lambda' + \varepsilon$.

For the method of finding x, y, z, V_x, V_y, V_z, see Appendix IX.

§ 348. The sign of c' in Eq. (110) determines the inclination to be *acute* or *obtuse*, and also the direction of the motion, the latter being *direct* when c' is positive, and *retrograde* when negative. The planet will be receding from or approaching the equinoctial according as z and V_z have the same or opposite signs, and it will be north or south of the equinoctial according as z is positive or negative. The signs of x and y will show in which quadrant the planet is projected on the plane of the equinoctial. See Appendix I.

§ 349. The position of the orbit is given in reference to the equinoctial; to obtain it in reference to the ecliptic is a mere operation of spherical trigonometry too obvious to require explanation.

§ 350. The disturbing action of the planets upon one another causes the nodes, inclinations, eccentricities, and perihelions to vary. The mean rate of change in each case is found by dividing the whole change, as ascertained at epochs widely separated, by the interval.

§ 351. The periodic time in mean solar days is found by multiplying the tabular periodic time, which is expressed in that of the earth as unity, by 365.24; and the mean distance in miles will be given by the product of the tabular distance into 95,000,000.

§ 352. *Dimensions and Geocentric Distances.*—Denote by X, Y, and Z the co-ordinates of the earth; by x, y, and z, those of a planet, referred to the centre of the sun; and by D the distance of the planet from the earth. Then

$$D = \sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2} \quad . \; . \; . \; . \quad (120)$$

§ 353. The horizontal parallax of any body is the apparent semi-diameter of the earth as seen from the body. Let π be the horizontal parallax of the sun, P' that of the planet, and r the radius vector of the earth; then, as the apparent semi-diameter of the earth is inversely proportional to the distance from which it is viewed, will

$$\pi : P' :: \frac{1}{r} : \frac{1}{D}$$

whence

$$P' = \pi . \frac{r}{D} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (121)$$

and this in Eq. (29) gives

$$d = \rho . \frac{s}{P'} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (122)$$

in which s is the planet's apparent semi-diameter measured with the micrometer, d its real semi-diameter, and ρ the earth's equatorial radius; whence the diameter, surface, and volume of the planet become known.

§ 354. Mercury and Venus are called *inferior* planets, being lower or nearer to the sun than the earth; the others are called *superior* planets, because they are higher or more distant from the sun than the earth.

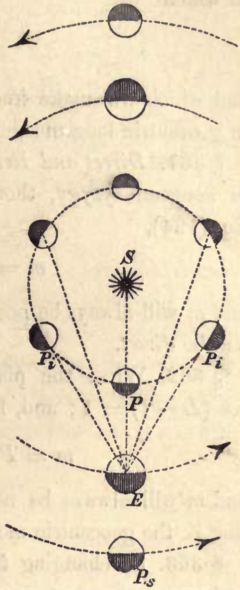
§ 355. When the geocentric longitude of a body is the same as that of the sun, the body is said to be in *conjunction;* when its longitude differs by 180°, in *opposition.* The superior planets may be in opposition, but the inferior planets never.

§ 356. A body in conjunction or opposition is also said to be in *syzygy.*

§ 357. When an inferior planet is in perigean syzygy, it is said to be in *inferior* conjunction; when in apogean syzygy, in *superior* conjunction.

§ 358. *Synodic revolution.*—The interval of time between two consecutive returns of a planet to apogean or perigean syzygy is called its *synodic revolution.*

Fig. 71.



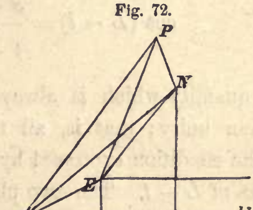
Denote by m the heliocentric mean daily motion of the earth in longitude; by n, that of any planet; and by T, the length of its synodic revolution; then will $m \sim n$ be the relative motion in longitude of the earth and planet, and

$$T = \frac{360^\circ}{m \sim n} \quad . \quad . \quad . \quad (123)$$

§ 359. *Geocentric Motion in Longitude.*—The angle at the earth, subtended by a body's linear distance from the sun, is called the body's *elongation;* the projection of a body's centre on the plane of the ecliptic, is called the *reduced place;* and the projection of its radius vector, is called the *curtate distance.*

Thus, let S be the sun, P a planet, E the earth, and PN a perpendicular from the planet to the plane of the ecliptic, intersecting the latter in N; then will SEP be the elongation, N the reduced place, and SN the curtate distance of the planet.

Fig. 72.



§ 360. Draw SV and EV to the vernal equinox; they will be sensibly parallel. Also draw $NN_{\prime}$ and $EE_{\prime}$ perpendicular to EV and SV, and make

$a = SN =$ mean curtate distance;
$\rho = EN =$ earth's distance from the reduced place;
$l = VSN =$ planet's heliocentric longitude;
$n =$ hourly change in the same;
$L = VSE =$ earth's heliocentric longitude;
$\lambda = VEN =$ planet's geocentric longitude;
$m =$ hourly change in the same.

Then, the mean distance of the earth from the sun being unity, will, Appendix X.,

$$m = P^2\left[a^2 + a^{\frac{3}{2}} - (a + a^{\frac{5}{2}}) \cdot \cos(L - l)\right] \cdot n \quad . \quad . \quad . \quad (124)$$

in which

$$P = \frac{\cos \lambda}{a \cos l - \cos L}$$

and which will make known the rate and direction of the body's motion in geocentric longitude.

§ 361. *Direct and Retrograde Motion; Stations.*—When the planet is in apogean syzygy, then will $L - l = 180°$, $\cos(L - l) = -1$; and, Eq. (124),

$$m = P^2 \cdot a \cdot (a + 1)(1 + a^{\frac{1}{2}}) \cdot n \quad . \quad . \quad . \quad . \quad . \quad (125)$$

and m will always be positive; that is, the geocentric motion of the planet will be *direct.*

§ 362. When the planet is in perigean syzygy, then will $L - l = 0$; $\cos(L - l) = 1$; and, Eq. (124),

$$m = P^2 \cdot a\,(a - 1)(1 - a^{\frac{1}{2}}) \cdot n \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

and m will always be negative, whether a be greater or less than unity; that is, the geocentric motion of the planet will be *retrograde.*

§ 363. In changing from direct to retrograde, and the converse, the body must appear stationary. This will make $m = 0$, and, Eq. (124),

$$\cos(L - l) = \frac{a + a^{\frac{1}{2}}}{1 + a^{\frac{3}{2}}} = \frac{1}{a^{\frac{1}{2}} + a^{-\frac{1}{2}} - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

a quantity which is always less than unity, whether a be greater or less than unity; that is, all the planets must sometimes appear stationary. The condition expressed by Eq. (127), may always be satisfied for two values of $L - l$. The two places of a body, in which it appears stationary, are called *stations.*

§ 364. Let the value of $L - l$ for one of the stations be φ; then, Eq. (124),

$$0 = P^2\left[a^2 + a^{\frac{3}{2}} - (a + a^{\frac{5}{2}}) \cos \varphi\right] \cdot n\,;$$

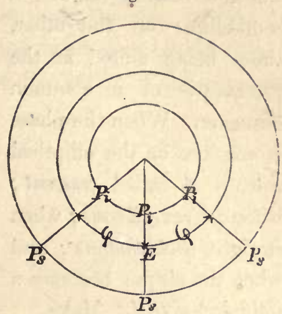
and subtracting from Eq. (124),

$$m = P^2 \cdot (a + a^{\frac{5}{2}})\left[\cos \varphi - \cos(L - l)\right] \cdot n \quad . \quad . \quad . \quad (128)$$

in which, as long as φ is less than $90°$, and $L - l$ greater than φ and less than $360° - \varphi$, m will be positive and the motion direct.

§ 365. Denote by n' the earth's mean motion in longitude; then will $n \sim n'$ be the mean relative heliocentric motion of the earth and planet; and denoting by t_r and t_d the durations of the retrograde and direct motions, we have

Fig. 73.



$$t_r = \frac{2\varphi}{n \sim n'} \quad . \quad . \quad (129)$$

$$t_d = \frac{360^\circ - 2\varphi}{n \sim n'} \quad . \quad . \quad (130)$$

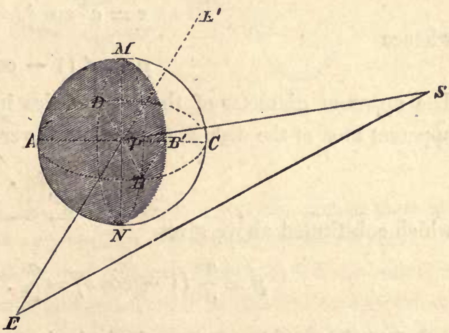
and the duration of the direct motion will be the longer.

It thus appears that in the course of one synodic revolution the planets appear sometimes to be stationary, then to move forward or in the order of the signs, then to be stationary again, and finally to move backwards.

§ 366. *Phases of the Planets.*—A body illumined by the sun, and shifting its place in reference to the sun and earth, presents to the latter different appearances at different times. These appearances are called *phases.*

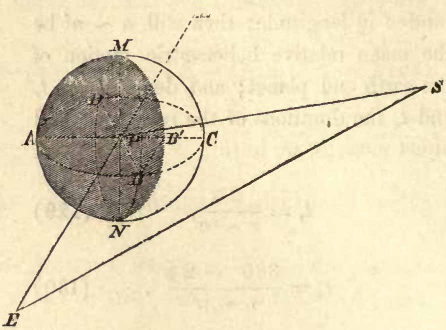
§ 367. To find the phase of a globular body, let S be the place of the sun's centre, E that of the earth, and P that of the body; $ADCB$ a section of the body by a plane through E, P, and S; a plane through P and perpendicular to PE will cut from the body's surface the section $AMCN$, which determines the hemisphere turned towards the earth; and another through P, and perpendicular to SP, will give the section $MDNB$, which determines the illuminated hemisphere turned towards the sun. The illuminated lower surface $MCNBM$ will be visible to the earth, and its projection on the plane $AMCN$ will give the shape and magnitude of the phase. The projection of the semicircle $MBNM$ will be a semi-ellipse $MB'NM$, of which the transverse axis is equal to the diameter of the body; its conjugate will vary with the angle which the projected plane makes with that of projection. The phase will therefore have for its boundary a semicircle on the

Fig. 74.



side towards the sun and a semi-ellipse on the other, these being united at the extremities of a common diameter. When the phase is concave on the elliptical side, it is called *crescent;* when convex, *gibbous;* when straight, *dichotomous;* and when the ellipse becomes a semicircle, *full.* Make

Fig. 74 bis.



$d =$ distance EP;
$a =$ apparent area of the semicircle $MCNM$ at distance unity,
$a' =$ " " " " distance d;
$p =$ " " phase $MCNB'M$ " "
$e =$ " " semi-ellipse $MB'NM$ " "
$\theta =$ angle $BPB' = SPE'$, the exterior angle of elongation.

Then

$$p = a' - e;$$

but

$$e = a' \cos \theta,$$

whence

$$p = a' (1 - \cos \theta).$$

The apparent diameter of the body varies inversely as the first, and the apparent area of the disk as the second power of the distance; whence

$$a' = \frac{a}{d^2},$$

which substituted above gives

$$p = \frac{a}{d^2} (1 - \cos \theta) = \frac{a}{d^2} \cdot \text{ver sin } \theta \quad . \quad . \quad . \quad . \quad (131)$$

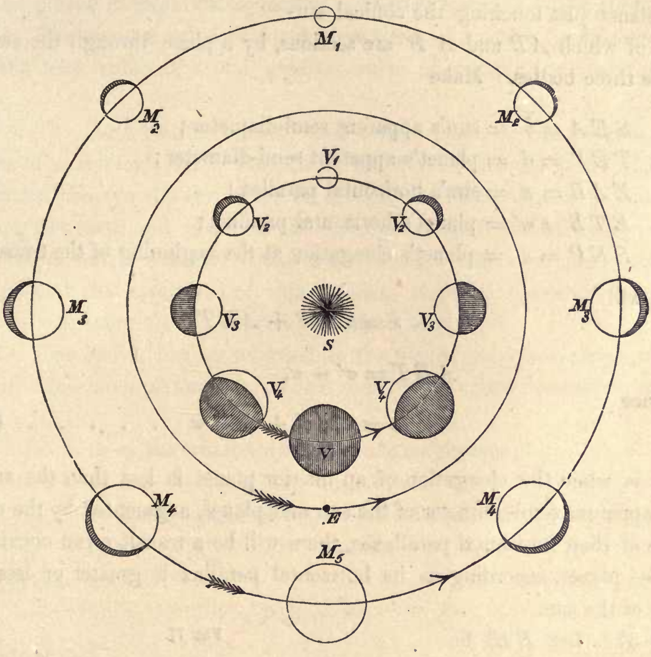
§ 368. The orbits of the principal planets have but slight inclinations to the ecliptic. At inferior conjunction of the inferior planets, the exterior angle of elongation will therefore approach to 0°, and the distance will be the least; at superior conjunction the exterior angle will be 180°, and the distance the greatest. In the first position the planet will be *invisible*, in the second *full*, and between these limits the phase will pass through crescent, dichotomous, and gibbous, with a continually decreasing diameter. From superior to inferior conjunction the same phases occur, but in the reverse order.

In the case of the superior planets, the exterior angle of elongation ap-

proaches to 180° both at conjunction and opposition, and it never can be as small as 90°. The phases of these bodies must, therefore, always be either gibbous or full; largest in opposition, and smallest in conjunction.

If S be the place of the sun; E that of the earth; V_1, V_2, &c., the

Fig. 75.



places of an inferior, and M_1, M_2, &c., those of a superior planet, then will these latter bodies exhibit the appearances represented in the figure.

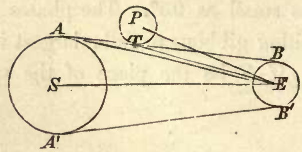
§ 369. *Transits, Occultations, and Transit Limits.*—A body which interposes itself between the earth and some other body, so as to conceal any portion of the latter from view, is said to make a *transit;* the masked body is said to be *occulted*, and the phenomenon is called a *transit* or an *occultation*, according as we refer to the masking or masked body.

§ 370. The nodal lines of all the planets lying in the plane of the ecliptic, are crossed twice a year by the earth. If at the time of crossing the nodal line of an inferior planet, the latter be in or near inferior conjunction, there will be a transit, and the planet will appear as a dark circle on the solar disk.

§ 371. *To find the greatest elongation consistent with a Transit.*—Conceive a conical surface tangent to the sun and earth. When the planet

at inferior conjunction passes wholly or in part within this surface, there will be a transit visible from some place on the earth.

Fig. 76.



Let S be the sun, E the earth, and P the planet just touching the conical surface, of which AB and $A'B'$ are sections, by a plane through the centres of the three bodies. Make

$$SEA = \delta = \text{sun's apparent semi-diameter;}$$
$$TEP = d = \text{planet's apparent semi-diameter;}$$
$$EAB = \pi = \text{sun's horizontal parallax;}$$
$$ETB = \pi' = \text{planet's horizontal parallax;}$$
$$SEP = s = \text{planet's elongation at the beginning of the transit;}$$

then will

$$s = \delta + d + AET,$$

but

$$AET = \pi' - \pi,$$

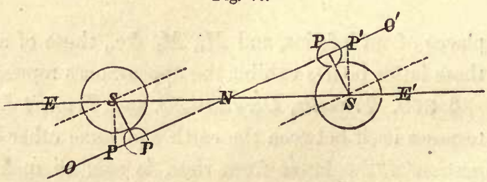
whence

$$s = \delta + d + \pi' - \pi \quad . \quad . \quad . \quad . \quad . \quad (132)$$

that is, when the elongation of an inferior planet is less than the sum of the apparent semi-diameter of the sun and planet, augmented by the difference of their horizontal parallaxes, there will be a transit or an occultation of the planet. according as its horizontal parallax is greater or less than that of the sun.

§ 372. Let EE' be an arc of the ecliptic, $O\,O'$ an arc of the planet's orbit, and N the node. Parallel to $O\,O'$, and at a distance from it equal to s, draw on either side a line cutting the ecliptic in S.

Fig. 77.



Now, if at the time of inferior conjunction the difference between the geocentric longitudes of the sun and node be less than SN, there must be a transit; if greater, there can be none. The distance SN is called a *transit limit.*

To find its value, make

$$SNP = i = \text{inclination of the planet's orbit;}$$
$$NS = l = \text{transit limit;}$$

then, in the right-angled triangle SPN,

$$\sin l = \frac{\sin s}{\sin i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (133)$$

The value of s is variable, being a function of the radii vectors of the earth and planet at inferior conjunction. The inclination i is also slightly variable. The greatest value of i and least value of s make l a *minimum* limit; the least value of i and greatest value of s make l a *maximum* limit.

§ 373. The earth returns sensibly to the same place of the heavens at intervals of a sidereal year. Any entire number of sidereal years which will contain the synodic revolution of a planet a whole number of times, will bring the earth and planet to the same places they simultaneously occupied before, and if a transit occur at one node, it will occur at the same node again at the expiration of this interval, provided the node be not carried by its proper motion beyond the transit limit.

§ 374. The bodies having returned to the places they previously occupied, will each have performed a whole number of entire revolutions, and making

$n =$ the number of the earth's revolutions;
$n' =$ " " planet's "
$P =$ the length of the earth's sidereal year;
$P' =$ " " planet's " "

we shall have

$$nP = n'P',$$

whence

$$\frac{n}{n'} = \frac{P'}{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

If P and P' be whole numbers, and the second member be reduced to its simplest terms, the numerator will be the interval in sidereal years between the consecutive transits at the same node, and this interval will be constant.

But if P and P' be not whole numbers, then will the numerators of the approximating fractions of the continued fraction, which give the values of the second member within the transit limits, be the variable intervals, in sidereal years, between the transits at the same node.

§ 375. *Masses and Densities of the Planets.*—The masses of such of the planets as have satellites may easily be found by the process of § 313, as soon as the periodic time of the planet and that of its satellite are determined by observation. But for such as have no satellites, recourse is had to a different process, which can be here indicated only in outline. A

planet undisturbed by the action of the others, would describe accurately its elliptical orbit about the common centre of inertia due to its own mass and that of the sun; and from the elliptical elements already described, its future places are, as we shall see, predicted with the greatest precision. The difference between these places and those actually observed, give the effects of the disturbing action of the other planets. To compute these effects, what are called *perturbating functions* are constructed upon the principles of mechanics. The masses of the perturbating or disturbing bodies enter these functions; and from the observed amount of perturbations the value of the masses are computed. *An. Mec.*, § 203,

§ 376. The masses and volumes being known, the densities result from the process of § 314.

§ 377. *Rotary motions.*—All the planets whose surfaces exhibit through the telescope distinct marks, are found to have a rotary motion in the same direction as those of the sun and earth, viz., from west to east.

§ 378. *Planetary Atmosphere.*—The existence of an atmosphere about a planet is indicated by the apparent displacement it occasions in the geocentric place of a star by refracting its light, when, by the motion of the earth and planet, the latter comes near the line of the star and observer.

The atmosphere about a planet is in fact a vast spherical lens, of which the central part is deprived of its transparency by the opaque materials of the planet, but of which the outer portion is free from obstruction and acts upon the light which passes through it with an energy due to its refractive power and density.

The height of the atmosphere is inferred from the greater or less angular distance between the star and planet when the displacement begins; and the density, which must be regulated by the same laws that govern the equilibrium of heavy elastic fluids upon the earth, from the amount of displacement.

§ 379. In detailing the physical peculiarities of the planets, their mean distances and times of sidereal revolutions, although contained in the synoptical table of elements, will be repeated; and in all cases in which dimensions or measures are given, they must be understood as expressed in the corresponding elements of the earth as unity. Thus, if it be the mean distance, density, volume, solar heat and light, sidereal day, &c., those of the earth are the respective units.

MERCURY.

§ 380. Proceeding outwards from the sun, Mercury is the first known planet. His mean distance is 0.3870985; sidereal year, 0.2408; true diameter, 0.398; volume, 0.063; mass, 0.175; density, 2.78,—approaching that of gold; intensity of its attraction for a unit of mass on its surface, called *surface gravitation*, 1,15; solar heat and light, 6.68; time of rotation upon its axis, called sidereal day, 1.20833.

The eccentricity of his orbit being large, his greatest elongation varies from 16° 12′ to 28° 48′. The latter being his greatest apparent distance from the sun, he is generally lost to us in the light of that body, and it is difficult, therefore, to observe him. His arc of retrogradation varies from 9° 22′ to 15° 44′.

§ 381. When to the west of the sun he rises before, and when to the east he sets after that luminary. In the former position he is called a morning, and in the latter an evening star.

§ 382. The sun appears nearly seven times as large to the inhabitants of Mercury as to us; and on the supposition that the intensity of solar light and heat varies inversely as the square of the distance, the solar illumination and temperature on Mercury would be 6.68 times that on the earth, as above. Heat and cold are, however, but relative terms, depending upon physical conditions as well as distance, and the Mercurian surface may be as cold as the earth's: the frosty summits of the Himalayas are nearer to the sun than the scorching plains of Hindostan.

§ 383. The changes of seasons on Mercury, depending, as they do, upon the inclination of his axis to that of his orbit, which has not been well determined, are not accurately known. If, as there are reasons to believe, this inclination have any considerable value, the mutations of Mercury's seasons must be very great; his tropical year being only about one-fourth that of the earth, his seasons, if they follow the same proportion, can only be of some two or three weeks' duration.

§ 384. Mercury's nodes are, and will for ages continue, in that part of the ecliptic which the earth passes in May and November, and his transits over the sun must occur in those months. His periodic time $= 87^d.97$, and that of the earth $= 365^d.256$, in Eq. (134), give the approximating fractions,

$$\frac{7}{29}; \quad \frac{13}{54}; \quad \frac{33}{137}; \quad \&c.$$

So that the intervals between the transits which may be expected at the same node are *seven*, *thirteen*, &c., years. The great inclination of Mercu-

ry's orbit makes his transit limits, Eq. (133), small, and the above intervals will not therefore always be those which separate the actual recurrence of the transits. The last transit occurred at the ascending node in 1848, the next will occur in 1861.

VENUS.

§ 385. Venus follows Mercury in the order from the centre. Her mean distance is 0.7233317; sidereal year, 0.6152; true diameter, 0.975; volume, 0.927; mass, 0.885; density, 0.95; surface gravitation, 0.93; solar heat and light, 1.91; sidereal day, 0.97315.

§ 386. Venus is the brightest of the planets, her light being of a brilliant white, and at times so intense as to cause a shadow. The elongations of her stations vary but little from 29°. Her phases are finely exhibited through the telescope. The southern horn of her crescent varies its shape, being alternately sharp and blunt, and the changes are attributed to the periodical interposition of high mountains by an axial rotation of Venus so as to intercept the solar light she at other times reflects to the earth from her southern surface. From these changes her sidereal day has been determined.

§ 387. Her axis is inclined to that of her ecliptic under an angle of 75°, thus placing her tropics at the distance of 15° from her poles, and her polar circles at the same distance from her equator. Her seasons succeed each other, therefore, very rapidly, there being two summers and two winters in each of her annual revolutions. Her atmosphere resembles in extent and density that of the earth.

§ 388. Her synodical revolution is 583.92 days. Venus is, therefore, about 292 days continuously to the east, and as long to the west of the sun. In the former position she sets after the sun, and is called an evening star; in the latter, she rises before the sun, and is called a morning star. Her greatest elongation is about 45°, and she is brightest when on her way from the east to the west of the sun, and at an elongation on either side of about 40°.

§ 389. The line of Venus's nodes lies in that part of the ecliptic through which the earth passes in June and December, and her transits occur in those months. The periodic time of Venus $= 224^d.7$, and that of the earth $= 365^d.256$, which, in Eq. (134), give the approximating fractions,

$$\frac{8}{13}, \quad \frac{235}{382}, \quad \frac{713}{1159}, \text{ \&c.}$$

and the transits at the same node may be expected at intervals of *eight*, *two hundred and thirty-five*, &c., years. Two transits, separated by an interval of eight years, will occur at one node, and then at the opposite node after an interval of one hundred and five, or one hundred and twenty-two years, between the last of the first pair and first of the second pair.

As astronomical phenomena the transits of Venus are of the highest importance. They afford the best means of ascertaining the sun's horizontal parallax, and therefore the earth's distance from the sun, and the dimensions of the solar system, expressed in terms of some known terrestrial measure.

§ 390. The principle on which the sun's horizontal parallax is found from a transit of Venus may be thus illustrated.

Conceive two observers situated at the opposite extremities *A* and *B* of the earth's diameter, which is perpendicular to the plane of the planet's orbit. To the observer *A*, the planet would appear to transit the sun's disk along the chord *m n*, and to the observer *B*, along the chord *p q*, being the intersections of the solar disk by two planes through the portion *D C* of Venus's orbit, described during the transit, and each observer. A third plane through the observers and Venus's centre would cut from the other two the lines *A a* and *B b*, and from the sun's disk the perpendicular distance *a b* between the chords. Now, because the angle *A V B* is equal to the angle *a V b*, *A B* will be to *a b* as Venus's distance from the earth is to her distance from the sun; that is § 385, as $1 - 0.723 : 0.723$, or as 1 to 2.61 nearly; and the radius of the earth, half of *A B* is to *a b*, as 1 to 5.22 nearly. The apparent magnitudes of two objects, viewed at the same distance, being directly proportional to the true magnitudes, the radius of the earth viewed at the distance of the sun, in other words, the sun's horizontal parallax, is equal to the angular distance between the chords divided by 5.22.

Fig. 78.

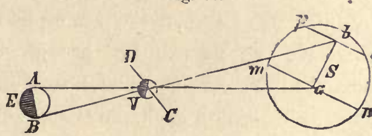
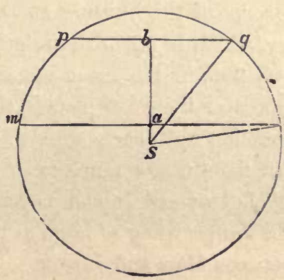


Fig. 79.



§ 391. The relative geocentric motion of the sun and planet into the *observed* durations of the transit at the two stations will give the chords *m n* and *p q*. The chords being known, as also the apparent

semi-diameters $S q$ and $S n$, the distances $S a$ and $S b$ become known, and therefore their difference $a b$.

§ 392. The general result of all the observations made on the transit of 1769 gives 8″.5776 for the sun's horizontal parallax. The next two transits of Venus will occur on Dec. 8th, 1874, and Dec. 6th, 1882.

MARS.

§ 393. Mars is the first of the superior planets. His mean distance is 1.5237; sidereal year, 1.8807; true diameter, 0.517; volume 0.1386; density, 0.95; equatorial gravitation, 0.493; solar heat and light, 0.43; sidereal day, 1.02694; oblateness, about 19; and the inclination of his axis to that of his ecliptic 30° 18′ 10″.8.

§ 394. He has a dense atmosphere of moderate height. His surface (Plate II., Fig. 2) exhibits through the telescope outlines of what are deemed to be continents and seas, the former being distinguished by a ruddy color, which is characteristic of this planet, and indicates an ochry tinge in the soil, contrasted with which the seas appear of a greenish hue.

These markings are not always equally distinct; and the variation is attributed to the formation of clouds and mists in the planet's atmosphere. Brilliant white spots sometimes appear at that pole which is just emerging from the long night of its polar winter, and are attributed to extensive snow-fields that push their borders to an average distance of some six degrees from either pole.

PLANETOIDS.

§ 395. Next to Mars come the class of small planets, which, on account of their comparatively diminutive size, are called *planetoids.* Little is known of them beyond their orbit elements, but they are interesting on account of their history and the speculations connected with their discovery, which began with the present century.

§ 396. If the mean distance of Mercury be taken from the mean distances of the other planets, the remainders will form a series of numbers doubling upon each other in proceeding outward from the sun. To this law there was a remarkable exception in the distance between the orbits of Mercury and Jupiter as compared with that between Mercury and Mars, the former being so large as to require the interpolation of another body between Mars and Jupiter.

§ 397. Although th• law is strictly empirical and wholly inexplicable

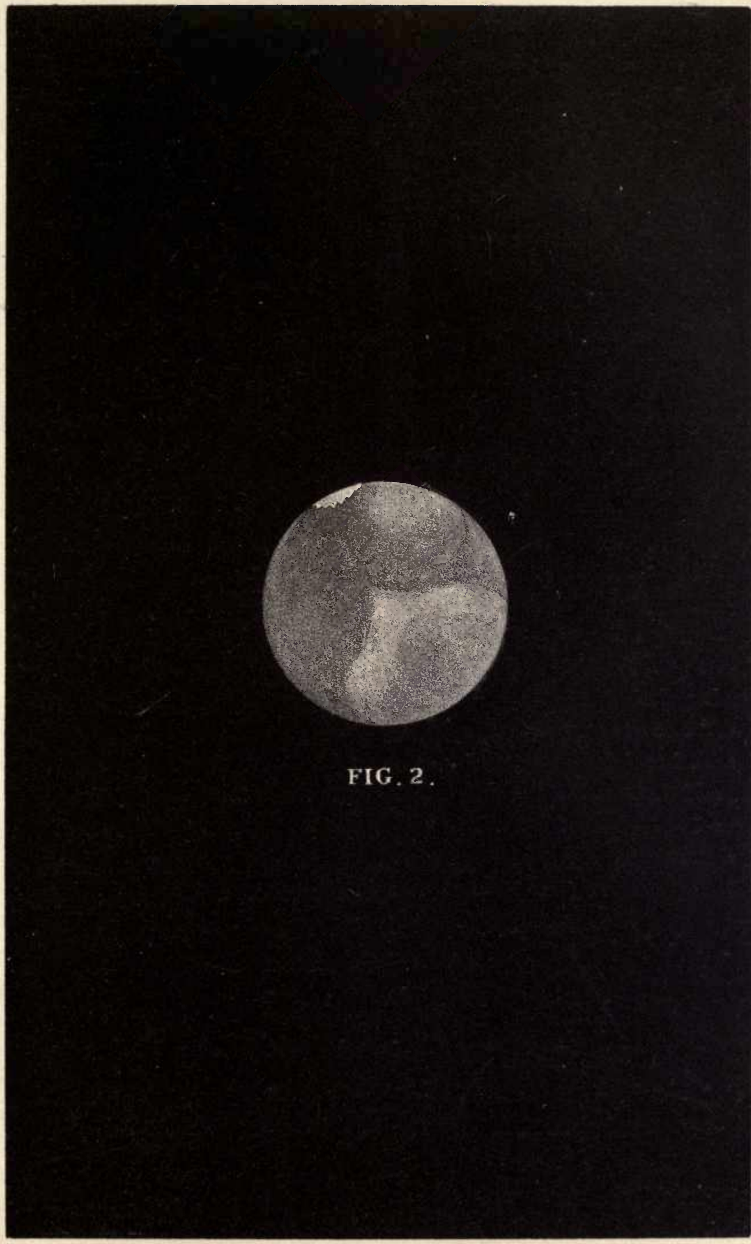


FIG. 2.

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a priori upon any known physical hypothesis, yet the coincidence was so remarkable as to induce the prediction that by proper search a planet would be found in the interpolated place.

§ 398. This body was only to be recognized by its proper motion. To detect this, an examination of the telescopic stars of the Zodiac was commenced, their places were carefully mapped, and on the first day of the present century, the prediction was verified by the addition of *Ceres* to the system. Her mean distance is 2.76692, and the hiatus was filled.

§ 399. But the discovery of Ceres was soon followed by that of Pallas, at the mean distance of 2.7728—nearly the same as that of Ceres—and the law was again broken.

§ 400. The points in which the paths of the new planets are intersected, on either side of the sun, by the line common to the planes of both orbits, are not very far apart, and it was suggested that Ceres and Pallas were but fragments of a larger planet that once revolved at an average distance, and which had been broken to pieces by some disruptive force. But where were the other fragments?

§ 401. A number of bodies projected in different directions from a common point, would each describe about the sun an hyperbola, a parabola, or an ellipse, depending upon the relations between the velocity of projection and the intensity of the sun's attraction upon the unit of mass, and in the case of elliptical orbits, the bodies would, abating the effects of the perturbating action of the other planets, return at fixed intervals to the place of departure.

§ 402. The opposite points of the heavens, in which the orbits of Ceres and Pallas approached most nearly each other, were therefore regarded as the common haunts of the suspected fragments, and the places especially to be watched, to detect their existence. A constant scrutiny of these points, and diligent revision of the maps of the zodiac, have resulted in the discovery, to the present time, of 91 of these little bodies.

§ 403. The mean distances of the planetoids vary about from 2.2 to 3.6, and periodic times about from 3.3 to 6.9. Their small size makes it difficult to determine their true dimensions, the diameter of the same individual, as given by the best authorities, varying from 0.02 to 0.20. They exhibit considerable variety of color; some have shown signs of possessing atmospheres, and those who regard them as debris of a single body, find evidence of an angular or fragmental figure in sudden changes of illumination, which have been observed, and which are attributed to the shifting of their bounding planes by a diurnal or axial rotation.

JUPITER.

§ 404. Jupiter is the largest, and except Venus, which he sometimes surpasses in this respect, the brightest of the planets. His mean distance is 5.202; sidereal year, 11.86; diameter, 11,2; volume, 1280.9; mass, 331.57; density, 0.24—but little greater than that of water; equatorial gravitation, 2.716; solar heat and light, 0.037; sidereal day, 0.41376; oblateness, 20; inclination of axis to that of his ecliptic, 3° 5′ 30″.

§ 405. The disk of Jupiter is always crossed, in a direction parallel to his equator, by dark bands or belts, presenting the appearance indicated in Plate III., fig. 3, which was taken by Sir John Herschel. These belts are not always the same, but vary in breadth and situation, though never in direction. They have sometimes been seen broken up and distributed over the whole face of the planet. From their parallelism to Jupiter's equator, their occasional variation and the appearance of spots upon them, it is inferred that they exist in the planet's atmosphere, and are composed of extensive tracts of clouds, formed by his trade-winds, which, from the great size of Jupiter, and the rapidity of his axial rotation, are much more decided and regular than those of the earth.

§ 406. The great oblateness of this planet is due to the shortness of his sidereal day, and its amount agrees with that assigned by theory to give him a figure of fluid equilibrium.

§ 407. From the small inclination of his axis to that of his ecliptic, there can be but little variation in the length of his days and nights, each of which is less than five of our hours; and changes of seasons must be almost, if not quite unknown to his inhabitants.

§ 408. Jupiter is attended in his circuit about the sun by *four* satellites or moons, which revolve about him from west to east, and present a miniature system analogous to that of which Jupiter himself is but a single individual, thus affording a most striking illustration of the effects of gravitation and of distance in grouping, as well as shaping the courses of the heavenly bodies. These satellites will be noticed under the head of *Secondary Planets.*

SATURN.

§ 409. Saturn is the next in order of size as he is of distance to Jupiter. His mean distance is 9.538850; sidereal year, 29.46; true diameter, 9.982; volume, 995.00; mass, 101.068; density, 0.102—little more than half that of water; equatorial gravitation, 1.014; solar heat and

FIG. 3.

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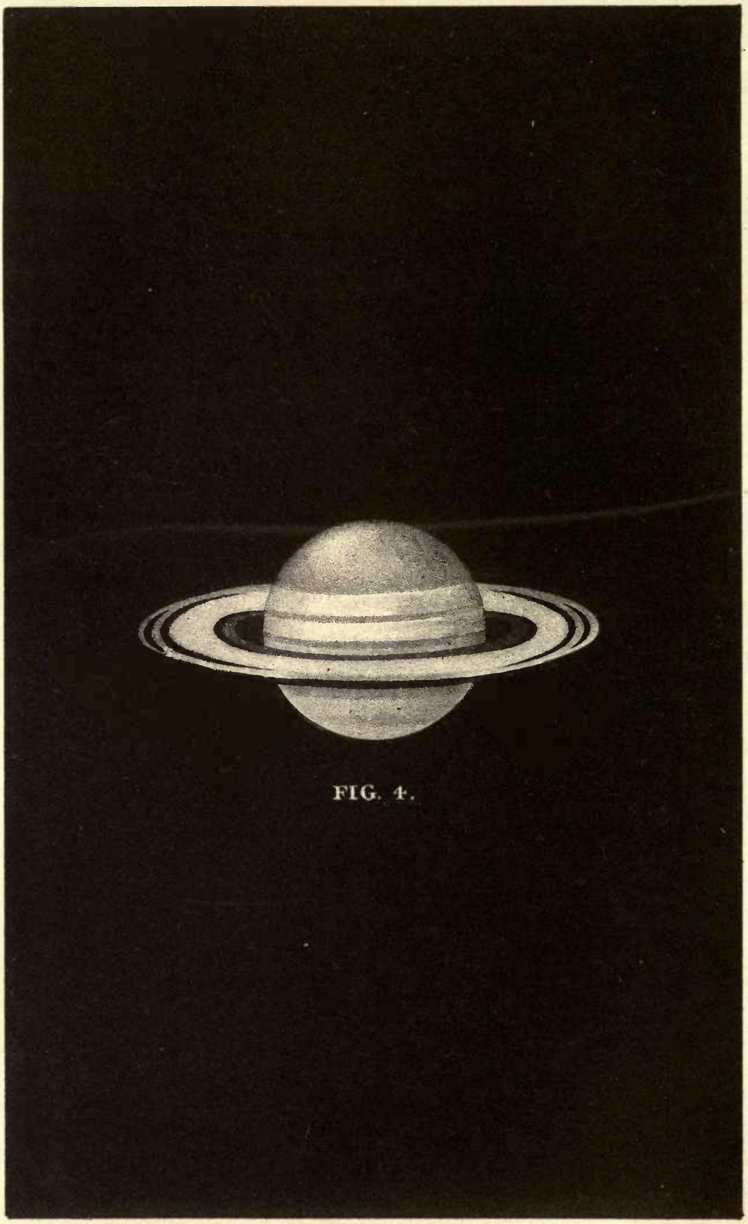

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light, 0.011; sidereal day, 0.43701; oblateness, 25; inclination of axis to that of orbit, 26° 49′, and to that of our ecliptic, 28° 11′.

§ 410. Saturn is the most curious and interesting body of the system, being attended by eight satellites or moons, and surrounded (Plate IV., Fig. 4), according to some authorities by two, and others by four, broad flat and extremely thin rings, concentric with each other and with the planet.

§ 411. The dimensions of the rings and planet, and the intervals as given by the advocates of but two rings, are,

	″	miles.
Exterior diameter of exterior ring	40.095 =	176,418
Interior " " "	35.289 =	155,272
Exterior diameter of interior ring	34.475 =	151,690
Interior " " "	26.668 =	117,339
Equatorial diameter of planet	17.991 =	79,160
Interval between the planet and interior ring	4.339 =	19,090
Interval between the rings	0.408 =	1,791
Thickness of ring not exceeding		230

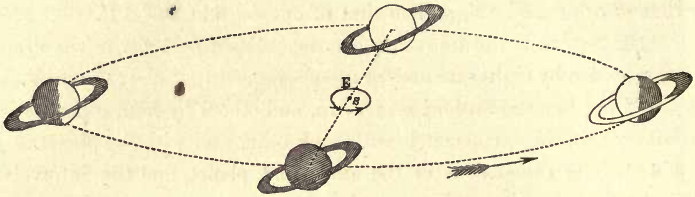
§ 412. The evidence of recent observations with very powerful instruments seems, however, in favor of a division of the outer ring, as just given, at a distance less than half its width from the exterior edge, and of the existence of a dusky ring still nearer the body of the planet, and composed of materials partially *transparent*, and possessing but feeble powers of reflection, resembling in these particulars a sheet of water. And there seem good reasons for believing that the rings are not precisely in the same plane.

The disk of the planet is crossed by parallel belts, similar to those of Jupiter; these are supposed to be due to Saturn's trade-winds. From the parallelism of the belts to the plane of the rings, it is inferred that the planet's axis of rotation is perpendicular to that plane, and this is confirmed by the occasional appearance of extensive dusky spots on his surface, which, when carefully watched, give the time of his rotation about an axis having that direction.

§ 413. By watching the different shades of illumination on different portions of the rings, the latter are found to complete a revolution in their own plane once in $10^h\ 32^m\ 15^s$, thus making their sidereal day 0.43906, which exceeds that of the planet itself by 0.00205.

§ 414. That the rings are opaque and non-luminous is shown by their throwing a shadow on the body of the planet on the side nearest the sun, and by the other side receiving that of the planet as shown in the figure.

Fig. 80.

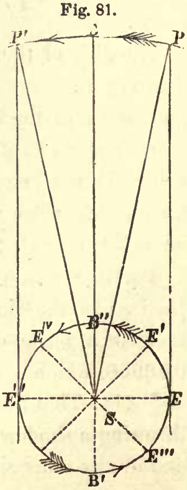


§ 415. The axes of the planet and rings preserve their directions unchanged during their orbital motion. The plane of the rings, which is inclined to that of the ecliptic under an angle of 31° 19′, intersects the latter plane in a line which makes with the line of the equinoxes an angle equal to 167° 31′, so that the nodes of the ring lie in longitudes 167° 31′ and 347° 31′.

§ 416. The orbital motion of the planet causes this intersection to oscillate, as it were, parallel to itself, in the plane of the ecliptic, through a distance on either side of the sun equal to the radius vector of Saturn's orbit; and the period of a semi-oscillation is one-half of the planet's period, or about 15 years. Within this period the plane of the ring must pass once through the sun, and from once to thrice through the earth, depending upon the initial position or place of the latter when the trace of the plane on the ecliptic touches the earth's orbit at the time of nearing the sun.

§ 417. Thus, let S be the sun, $E\,E'E''E'''$ the earth's orbit, $P\,P'$ an arc of Saturn's orbit projected upon the plane of the ecliptic, $P\,E$ and $P'\,E''$ the traces of the plane of the rings on the same, and tangent to the earth's orbit, and suppose the motion of the earth and of Saturn to take place in the direction indicated by the arrow-heads. Draw $S\,B$ parallel to $P\,E$ and $P'\,E''$, and make

Fig. 81.



$r = S\,P$ = the mean distance of Saturn;
$r' = S\,E$ = " " " of earth;
$\alpha = P\,S\,P'$ = the angle at the sun subtended by $P\,P'$:

then, since the angle $P\,S\,B = S\,P\,E$, we have

$$\sin \tfrac{1}{2}\,\alpha = \frac{r'}{r} = \frac{1}{9.54} = 0.1082,$$

whence $\alpha = 12°\ 2'$,

which divided by $2' \, 0''.6$, the mean motion of Saturn, gives 359.46 days, wanting only 5.8 days of a complete year; that is to say, the earth describes nearly one entire revolution in the time during which the earth's orbit is traversed by the plane of the ring.

§ 418. The rings are invisible when their plane passes between the sun and earth, their enlightened face being then turned from the latter body; and the interval of non-appearance will be that between any two epochs at which the plane passes the sun and earth, and of which the effect of one is to throw these bodies on opposite and the other to restore them to the same side of this plane.

§ 419. If the initial place of the earth be at E^{IV}, nearly three days in advance of B'', then will the plane itself pass the sun and earth at the same time, the earth being at B', and these bodies could not be on opposite sides of the plane of the rings during its present visit to the earth's orbit. If the initial position of the earth be at E', nearly three days in advance of E, it will be at E'' when the plane passes the sun; the rings will then disappear, and continue invisible till the earth meets and passes their advancing plane, which it will do somewhere in the quadrant $E''B'$; they will then reappear, and continue visible for the next fifteen years. If the earth's initial place be at E''', some days in advance of B', it will meet and pass the plane in the same quadrant, the rings will disappear and continue invisible till their plane is overtaken and passed again by the earth somewhere in the quadrant EB''; when the plane passes the sun the earth will be in the quadrant $B''E''$, and the rings will again disappear, and again become visible only when their plane is recrossed by the earth in the quadrant $E''B'$. Thus, with this initial place, the earth will cross the plane of the rings three times in one year, and there will be two disappearances.

§ 420. When the plane of the ring passes through the sun, the edge of the ring alone is enlightened, and can only appear as a straight line of light projecting from opposite sides of the planet in the plane of his equator, and parallel to his belts. This phase of the ring has been seen, but it requires the most powerful telescopes; and from the fact of its non-appearance in a telescope which would measure a line of light one-twentieth of a second in breadth, of which the subtense at Saturn's distance is 230 miles, it is inferred that the thickness of the ring cannot exceed this latter dimension.

§ 421. When the dark side of the ring is turned to the earth, the planet appears as a bright round disk with its belts, and crossed equatorially by a narrow and perfectly black line. This can only happen when

the planet is less than 6° 1′ from the node of his rings. Generally the northern side is enlightened when the heliocentric longitude of Saturn is between 172° 32′ and 341° 30′, and the southern when between 353° 32′ and 161° 30′. The greatest opening occurs when the heliocentric longitude of the planet is 77° 31′ or 257° 31′.

URANUS

§ 422. Uranus is one of the more recently discovered planets, being only recognized as a planet for the first time in 1781, though it had often been seen before and mistaken for a fixed star.

Of this planet nothing can be seen but a small round uniformly illuminated disk without rings, belts, or discernible spots. His mean distance is 19.18239; sidereal year, 84.01; true diameter, 4.36; volume, 82.91; mass, 14.25; density, 0.17; equatorial gravitation, 0.75; solar heat and light, 0.003. He is attended by six satellites, which will be noticed presently.

NEPTUNE.

§ 423. Neptune is the last known planet in the order of distance, and third in size. Its discovery dates only from 1846, though its existence had been suspected from certain irregularities in the motion of Uranus, which could only be attributed to the disturbing action of some body exterior to itself.

The departures of Uranus from places assigned by the combined action of the known bodies of the system, and certain assumed conditions in regard to position and shape of orbit, direction of motion, and mean distance, rendered highly probable by analogy, were the data from which, by the methods of physical astronomy, was wrought out in the closet in Paris, the place of a new planet whose disturbing action would account for the unexplained waywardness of Uranus. The result was sent to an observer in Berlin, and in the evening of the very day of its receipt in the latter city, Neptune was added to the known system by actual observation. It was found within 52′ of the place assigned, and its discovery, in all its circumstances, must ever be regarded as one of the greatest triumphs of modern science.

§ 424. Neptune's mean distance is 30.0367; periodic time, 164.6181; real diameter, 4.5; volume, 91.125; mass, 18.219; density, 0.208; equatorial gravitation, 0.9035; solar heat and light, 0.0011.

The apparent size of the sun as seen from the earth, bears to that as seen

from Neptune, about the relation of an ordinary orange to a common duck-shot.

§ 425. Neptune has at least one satellite, and certain appearances have indicated a second, and also a ring, but of these there are yet doubts.

General Remark.

§ 426. In the foregoing enumeration of the physical peculiarities of the planets, one is impressed by the great differences in their respective supplies of heat and light from the sun; in the relations which the inertia of matter bears to its weight at their surfaces; and in the nature of the materials of which they are composed, as inferred from variety of mean density. The intensity of solar radiation is nearly seven times greater on Mercury than on the earth, and on Neptune 900 times less, giving a range of which the extremes have the ratio of 6300 to 1. The efficacy of weight in counteracting muscular effort and repressing animal activity on the earth, is less than half that on Jupiter, more than twice that on Mars, and probably more than twenty times that on the planetoids, making a range of which the limits are as 40 to 1. Lastly, the density of Saturn does not exceed that of common cork. Now, under the various combinations of elements so important as these, what an immense diversity must exist in the conditions of animal life, if the planets, like our earth, which teems with living beings in every corner, be inhabited! A globe whose surface is seven times hotter than ours or 900 times colder, on which a man might by a single muscular effort spring fifty feet high, or with difficulty lift his foot from the ground; where his veins would burst from deficiency or collapse from excess of atmospheric pressure, affords to our ideas an inhospitable abode for animated beings. But we should remember that heat and cold, light and darkness, strength and weakness, weight and levity, are but relative terms; and to the very conditions which convey to our minds only images of gloom and horror, may be adjusted an animal and intellectual existence which make them the most perfect displays of wisdom and beneficence.

SECONDARY BODIES.

§ 427. The secondary bodies are those which revolve about the planets, and accompany them around the sun. Of these, *twenty* are known at the present time. *One* belongs to the earth, *four* to Jupiter, *eight* to Saturn. *six* to Uranus, and *one* to Neptune. They are commonly called satellites, and sometimes *moons*, but this latter appellation is more particularly applied to the earth's secondary.

THE MOON.

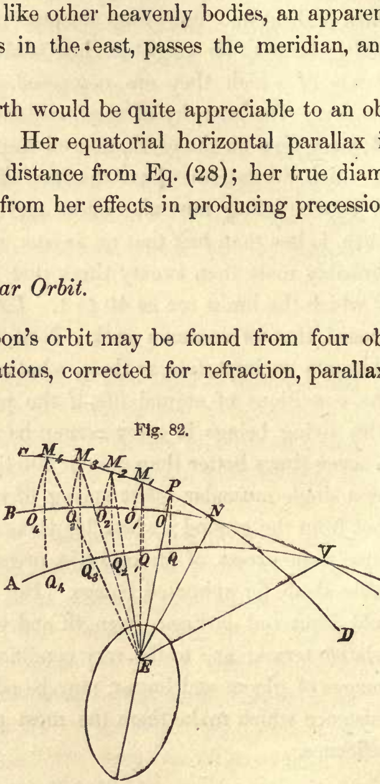
§ 428. The moon revolves in an elliptical orbit, of which one of the foci is at the earth's centre. Its motion is from west to east, and its angular velocity about the earth is much greater than that of the earth around the sun. The moon appears, therefore, to move among the fixed stars in the same direction as the sun, but more rapidly; and from the axial motion of the earth she has, like other heavenly bodies, an apparent diurnal motion, by which she rises in the east, passes the meridian, and sets in the west.

§ 429. The oblateness of the earth would be quite appreciable to an observer at the distance of the moon. Her equatorial horizontal parallax is therefore found from Eq. (24); her distance from Eq. (28); her true diameter from Eq. (29); and her mass from her effects in producing precession and nutation.

Lunar Orbit.

§ 430. The elements of the moon's orbit may be found from four observed right ascensions and declinations, corrected for refraction, parallax, and semi-diameter.

Fig. 82.



Let $D\ C$ be an arc in which the plane of the orbit cuts the celestial sphere; VB an arc of the ecliptic, and VA of the equinoctial; V the vernal equinox, N the ascending node, P the perigee, and M_1, M_2, M_3, M_4 the geocentric places of the moon.

First convert the geocentric right ascensions and declinations into geocentric longitudes and latitudes, and make

$$
\begin{aligned}
\nu &= VN &&= \text{longitude of node;} \\
i &= CNB &&= \text{inclination of orbit;} \\
l_1 &= V\,O_1 &&= \text{longitude of } M_1; \\
l_2 &= V\,O_2 &&= \quad\text{"} \quad M_2; \\
\lambda_1 &= M_1\,O_1 &&= \text{latitude of } M_1; \\
\lambda_2 &= M_2\,O_2 &&= \quad\text{"} \quad M_2.
\end{aligned}
$$

then in the right-angled triangles $M_1 N O_1$ and $M_2 N O_2$, we have

$$\left.\begin{array}{l}\sin(l_1 - \nu) = \cot i \, . \tan \lambda_1 \\ \sin(l_2 - \nu) = \cot i \, . \tan \lambda_2\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (135)$$

and by division

$$\frac{\sin(l_1 - \nu)}{\sin(l_2 - \nu)} = \frac{\tan \lambda_1}{\tan \lambda_2}.$$

Adding unity to both members, reducing to common denominator, then subtracting each member from unity, reducing as before, and finally dividing one result by the other, we find

$$\frac{\sin(l_2 - \nu) + \sin(l_1 - \nu)}{\sin(l_2 - \nu) - \sin(l_1 - \nu)} = \frac{\tan \lambda_2 + \tan \lambda_1}{\tan \lambda_2 - \tan \lambda_1};$$

replacing the members by their equals, we have

$$\tan\left[\frac{l_2 + l_1}{2} - \nu\right] = \tan \tfrac{1}{2}(l_2 - l_1) \, . \frac{\sin(\lambda_2 + \lambda_1)}{\sin(\lambda_2 - \lambda_1)} \quad . \qquad (136)$$

Also, from first of Eqs. (135), we have

$$\cot i = \frac{\sin(l_1 - \nu)}{\tan \lambda_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (137)$$

whence ν and i are known.

The longitude of the ascending node, increased by the angular distance of a body from the same node, is called the *Orbit Longitude.*

Make

$$\begin{array}{rll}
v_1 = & VEN + NEM_1 = & \text{orbit longitude of } M_1; \\
p = & VEN + NEP \;\; = & \text{“} \quad \text{“} \quad \text{perigee}; \\
\varphi = & PEM_1 = v_1 - p = & \text{true anomaly of } M_1; \\
e = & & \text{eccentricity of orbit}; \\
m = & & \text{mean motion of moon in orbit}; \\
t_1 = & & \text{time since epoch for } M_1; \\
L = & & \text{mean orbit longitude at epoch.}
\end{array}$$

Then resuming Eq. (48), we have

$$L + m t_1 = v_1 - 2e \sin(v_1 - p) \quad . \quad . \quad . \quad . \qquad (138)$$

in which

$$v_1 = \nu + \tan^{-1} \frac{\tan(l_1 - \nu)}{\cos i} \quad . \quad . \quad . \quad . \qquad (139)$$

Four values for the geocentric longitudes denoted by l_1, l_2, l_3, l_4, in Eq. (139), give four values for v, viz., v_1, v_2, v_3, and v_4; and these, and the times

of observation t_1, t_2, t_3, and t_4, in Eq. (138), give four equations involving the four unknown quantities L, m, e, and p; whence these become known precisely as in § 197, employing for the purpose Eqs. (50), (51), (53), and (54).

§ 431. Denoting the ecliptic longitude VO of the perigee by p_1, we have, in the triangle NPO, right-angled at O,

$$\tan NO = \tan (p - \nu) \,.\, \cos i,$$

and

$$p_1 = \nu + \tan^{-1} [\tan (p - \nu) \,.\, \cos i] \quad . \quad . \quad . \quad (140)$$

§ 432. In the same way, denoting the mean ecliptic longitude of the moon at the epoch by L_1,

$$L_1 = \nu + \tan^{-1} [\tan (L - \nu) \,.\, \cos i] \quad . \quad . \quad . \quad . \quad (141)$$

§ 433. The passage of the moon through one entire circuit of 360° around the earth, is called a *sidereal revolution.* The interval of time required to perform a sidereal revolution is called a *sidereal period.* Denote the sidereal period by s, then will

$$s = \frac{360^\circ}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (142)$$

The equation of the orbit, the centre of the earth being the pole, is

$$r = \frac{a\,(1 - e^2)}{1 + e \cos (v - p)};$$

and the value of r being found by means of Eq. (28), that of the mean distance a will result, and every thing in regard to the moon's path becomes known.

§ 434. At the epoch January 1st, 1801, the elements of the lunar orbit were

Mean $a = 59.96435000$ of the earth's equatorial radius;
" $s = 27.321661418$ mean solar days;
" $e = 0.054844200$;
" $\nu = 13^\circ\ 53'\ 17''.7$;
" $p_1 = 266^\circ\ 10'\ 07''.5$;
" $i = 5^\circ\ 08'\ 47''.9$;
" $L_1 = 118^\circ\ 17'\ 08''.3$.

§ 435. The moon's true diameter, Eq. (29), is 0.27280, or about 2153 miles; volume, 0.0204; mass, 0.011399; density, 0.5657; and surface gravitation, 0.1666.

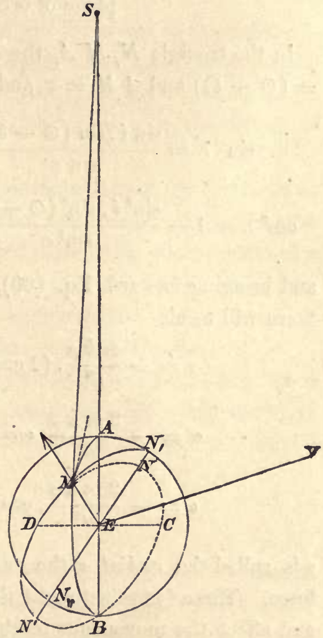
§ 436. Comparing the lunar elements which depend upon the orbit as

determined at different times, they are all found to vary. The nodes have a retrograde and the perigee a direct motion, the former performing a complete revolution in 18.6, and the latter in 8.854 years. The inclination fluctuates between 4° 57′ 22″ and 5° 20′ 06″; the mean distance has a secular variation, and it is at the present time diminishing; the same is true of the sidereal revolution, and the mean motion of the moon is increasing. All these changes are due to the disturbing action of the other bodies of the system, but principally of the sun. The action of the protuberant ring of matter about the equator of the earth also has its effect.

Disturbing Forces.

§ 437. To illustrate the way in which these changes of the lunar orbit are brought about, let E be the earth, S the sun, M the moon, moving in her orbit in the direction $MDN'N$; N and N' being the nodes, and $E\,V$ the direction of the vernal equinox. Then, resuming equations (80) and (81), making $\rho = EM$, the radius vector of the moon, and employing in all other respects the notation of § 286, ν becomes the change which the sun's attraction causes in the weight of a unit of the moon's mass due to the earth's attraction, and τ the change which the sun's attraction causes in the force normal to the radius vector and in the plane passing through the sun, earth, and moon. This latter force being in general oblique to the plane of the lunar orbit, urges the moon out of that plane, and causes her to describe a curve of double curvature, while the former has no such action.

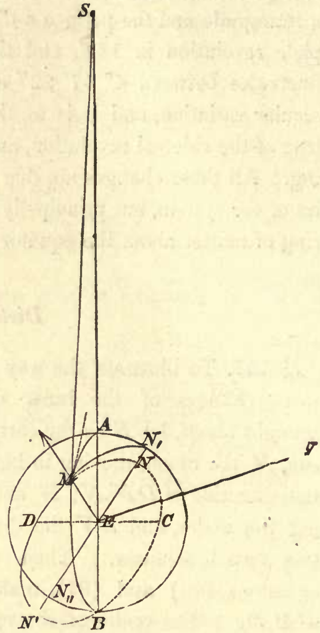
Fig. 86



Resolve τ into two components, one perpendicular to the radius vector and in the plane of the orbit, the other normal to this latter plane. For this purpose conceive a sphere of which the centre is at that of the earth, and radius, the radius vector $\rho = EM$, of the moon. Its surface will be cut by the plane of the ecliptic in $AN_{,}BN_{,,}$, by that of the lunar orbit

in $N_{,} M N_{,,}$, and by that of the sun, earth, and moon in $A M B$. Make

Fig. 83 bis.



$\odot = V E S =$ sun's longitude;
$\Omega = V E N_{,} =$ long. of moon's node;
$i = M N_{,} A =$ inclination of lunar orbit;
$\lambda = N_{,} M A =$ inclination of orbit to plane of sun, earth, and moon;
$\pi = \tau . \cos\lambda =$ component of τ in plane of lunar orbit;
$o = \tau . \sin\lambda =$ component normal to plane of orbit.

In the triangle $N_{,} M A$, the arc $N_{,} A = (\odot - \Omega)$ and $A M = \varphi$, and we have

$$\sin\lambda = \frac{\sin i . \sin(\odot - \Omega)}{\sin\varphi};$$

$$\cos^2\lambda = 1 - \frac{\sin^2 i . \sin^2(\odot - \Omega)}{\sin^2\varphi};$$

and bringing forward Eq. (80), and replacing τ by its value in Eq. (81), there will result

$$\nu = -\frac{m k \rho}{d^3} . (2\cos^2\varphi - \sin^2\varphi) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (143)$$

$$\pi = -\frac{2 m k \rho}{d^3} . \cos\varphi . \sqrt{\sin^2\varphi - \sin^2 i \sin^2(\odot - \Omega)} \quad (144)$$

$$o = -\frac{2 m k \rho}{d^3} . \cos\varphi . \sin i . \sin(\odot - \Omega) \quad . \; . \; . \; . \quad (145)$$

ν is called the *radial*, π the *transversal*, and o the *orthogonal* disturbing force. These forces acting at right angles are independent of each other, and affect the movements of the moon in modes perfectly distinct, which will become manifest by discussing the above equations.

§ 438. The radial force being directed towards or from the earth's centre, changes the simple law of gravitation, and alters the elliptic form of the orbit, sometimes diminishing and sometimes increasing its eccentricity, and shifting the place of its greatest curvature, that is, the position of the apsides.

§ 439. The transversal force is exerted to accelerate or retard the

moon's motion in her orbit, and to give rise to fluctuations to and fro about that due to the action of the earth alone, and thus to alter the elliptic path.

§ 440. The orthogonal force deflects the moon from the plane in which she would move under the undisturbed action of the earth, and causes her to describe a curve of double curvature. A plane through two consecutive elements of such a path must in general be oblique to that through two other consecutive elements, and these two planes must in general intersect the plane of the ecliptic in different lines; that is, the orthogonal force is effective in producing a motion of the nodes. By discussing the value of this force, it will be found that while it causes the nodes to move in different directions at different times, on the whole, it causes them to retrograde.

§ 441. Nothing has thus far been said of the variations in these disturbing forces arising, all other things being equal, from the change in the value of d, or the earth's distance from the sun. This gives rise to a still further complication by introducing an annual variation in the values of v, π, and υ.

§ 442. The other bodies of the system produce effects similar to those of the sun, but much less in degree. The protuberant ring of matter which projects beyond the sphere described upon the earth's polar axis has, as already remarked, its effect also; so that the longitude of the moon, as determined from the elements of a true elliptic motion, must receive from 30 to 40 corrections to obtain that of her true place.

§ 443. These corrections are called equations; their forms are determined by investigations in physical astronomy, and their coefficients are computed from the observed departures of the actual from the elliptic places.

Librations.

§ 444. The moon revolves uniformly about an axis inclined to that of her orbit, under an angle of 6° 38′ 58″, which is slightly variable; and the time of one revolution is equal to her sidereal period.

§ 445. Were her orbital motion uniform, and her axis perpendicular to her orbit, this equality would cause the moon always to present the same face to the earth. As it is, however, the visible portion of her surface is slightly variable, and in the course of a sidereal period we see a little more than a hemisphere. The changes of orbital motion cause small portions of her surface near her eastern and western borders to enter and depart from the field of view in the course of each revolution; and the inclination of

her axis to that of her orbit exposes to us her north or south pole alternately within the same period. These circumstances give rise to apparent oscillatory motions in the moon itself, which are called *librations;* those due to irregularity of orbital motion are called *longitudinal*, and those which arise from inclination of axis, *latitudinal librations.*

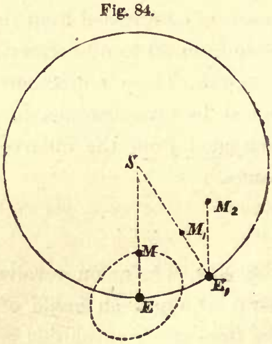
§ 446. In addition, slight variations take place in the visible portions of the moon's surface from changes in the observer's point of view, by the earth's rotation. These are called *parallactic librations.*

Lunar Periods.

§ 447. The moon's equinoctial is inclined to the ecliptic, and its ascending node always exactly coincides with the descending node of her orbit; so that the moon's axis describes a conical surface about the axis of the ecliptic once in 18.6 years.

§ 448. The passage of the moon from conjunction with the sun to conjunction again, or from opposition to opposition, is called a *synodic revolution.* Her passage from one longitude to the same longitude again, a *tropical revolution;* from perigee to perigee, or from apogee to apogee, an *anomalistic revolution;* from one node to the same node again, a *nodical revolution.* The intervals of time required to perform these revolutions are called *periods.*

Fig. 84.



§ 449. To find the length of either of these periods, say the synodic, let S be the sun, E the earth, M the moon in conjunction, E_1 the place of the earth at the next conjunction of the moon, then at M_1. Draw E_1M_2 parallel to $E\,S$. At the second conjunction the moon will have revolved through 360° about the earth, increased by the angle $M_2\,E_1\,M_1 = E\,S\,E_1$ = the earth's angular motion in the same time. Make

$$m = \text{moon's mean daily motion;}$$
$$n = \text{earth's} \quad \text{"} \quad \text{"} \quad \text{"}$$
$$t = \text{synodic period.}$$

Then
$$t\,m = 360° + E\,S\,E_1,$$
$$t\,n = \qquad E\,S\,E_1,$$

and by subtraction,

$$t\,m - t\,n = 360°;$$

whence $$t = \frac{360^\circ}{m - n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (146)$$

§ 450. Here n denotes the real angular motion of the earth, which is equal to the apparent angular motion of the sun. If it be replaced by the apparent geocentric motion of the vernal equinox, that of the apogee, or that of the node, taking care to give to each its appropriate sign (plus when the motion is direct and negative when retrograde), the corresponding period will result. The mean daily motion of the vernal equinox is equal to $50''.2$ divided by $365^d.242+$; that of apogee to 360°, divided by the number of mean solar days in 8.854 years; and that of the node by the number of days in 18.6 years.

The synodic period of moon			= 29.53 +	mean solar days.	
The anomalistic "		"	= 27.55 +	"	"
The tropical	"	"	= 27.32 +	"	"
The nodical	"	"	= 27.21 +	"	"

The synodic period of the moon is called a *lunar month*, or *lunation*.

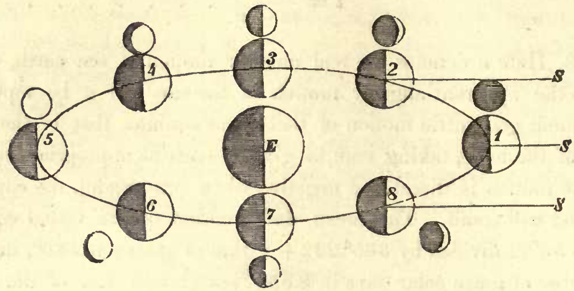
Lunar Phases.

§ 451. The sun's distance from the earth being 23984, and that of the moon only 59.96 times the earth's radius, the angle at the sun subtended by the semi-transverse axis of the lunar orbit is $0^\circ\ 08'\ 36''$; so that rays of light proceeding from the sun to the moon and earth may be regarded as sensibly parallel; and the exterior angle of elongation $S\,P\,E'$, Fig. 74, and Eq. (131), may be assumed equal to the true elongation $S\,E\,P$. Also the variation in the moon's distance is too small to produce sensible change in her apparent diameter to the naked eye, and the change becomes perceptible only when viewed through measuring instruments. The apparent diameter varies from $29'\ 21''.91$ to $33'\ 31''.07$, that at the mean distance being $31'\ 07''$.

§ 452. Resuming Eq. (131), making d constant, and δ equal to the moon's elongation, and supposing the sun to the right of the figure in the direction of $E\,S$ produced, the earth at E, and the moon successively in the positions 1, 2, 3, 4, 5, 6, 7, 8, we shall find the phases represented in the figure on next page.

When in conjunction at 1, δ or the elongation is zero, the moon is invisible, and this phase is called *new moon*. When at 2, the elongation being 45° east, the moon is said to be in *first octant*, and the phase is

Fig. 85.



crescent. When at 3, the elongation being 90° east, the moon is said to be in *first quarter*, and the phase is *dichotomous.* When at 4, the elongation being 135° east, the moon is said to be in *second octant*, and the phase is *gibbous.* When in opposition at 5, the elongation is 180°, the phase is full, and is called *full moon.* When at 6, the elongation being 135° west, the moon is said to be in *third octant*, and the phase is *gibbous.* When at 7, the elongation being 90° west, the moon is said to be in the *third quarter*, and the phase is again *dichotomous.* When at 8, the elongation being 45° west, the moon is said to be in *fourth octant*, and the phase is *crescent.* The interval of time required for the moon to pass through all these phases and resume them anew, is one synodic period, or *lunation.*

§ 453. The earth presents to the moon the same phases that the moon does to us; the angle of elongation of the earth, as seen from the moon, being always the supplement of the elongation of the moon, as seen from the earth.

§ 454. The pale light of the moon, by which its outline is defined in conjunction, is due to the light reflected from the earth, then full, falling upon the dark side of the moon

ECLIPSES OF THE SUN AND MOON.

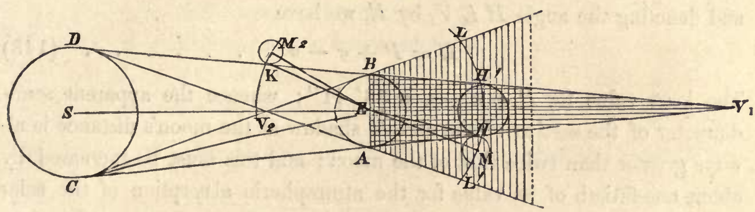
§ 455. The planets and satellites, being opaque, non-luminous bodies, and receiving their light from the sun, which is of vastly greater size, cast conical shadows, of which the surfaces produced must always be tangent to the sun's surface. The axes of the shadows cast by the planets, lie in the planes of their respective orbits. That of the earth is in the plane of the ecliptic; and if at the time of syzygy the moon be near one of her nodes, she will either pass within the luminous portion of the conical

space between the earth and sun, or enter the earth's shadow, according as her phase is *new* or *full.*

In the first case, she will mask the whole or part of the sun from some portions of the earth's surface; and in the latter, will suffer a loss of the light she herself receives from that body.

§ 456. The obscuration of the sun, by the interposition of the moon between the sun and earth, is called a *solar eclipse.* The obscuration of the moon, by a loss of solar illumination while within the earth's shadow, is called a *lunar eclipse.*

Fig. 86.



§ 457. Let S be the sun, E the earth, $D\,V_1$ and $C\,V_1$ tangents to the sun and earth; $A\,V_1\,B$ will be the earth's shadow. Let M be the moon just entering the shadow, and $H\,H'$ a right section of the latter at the distance of the moon. Make

$$\begin{aligned} \pi &= E\,C\,A && = \text{sun's horizontal parallax;} \\ \sigma &= C\,E\,S && = \text{sun's apparent semi-diameter;} \\ P &= E\,H\,A && = \text{moon's equatorial horizontal parallax;} \\ s &= H\,E\,M && = \text{moon's apparent semi-diameter;} \\ R &= E\,A && = \text{earth's equatorial radius;} \end{aligned}$$

then in the triangle $E\;V_1\;C$,

$$\text{angle } V_1 = \sigma - \pi;$$

in the triangle $E\,H\,V_1$,

$$\text{angle } H = 180^\circ - P;$$

and same triangle,

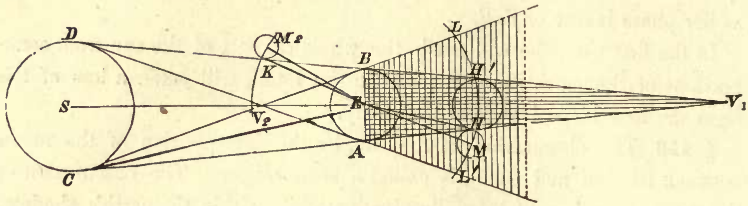
$$E\,V_1 : EH :: \sin(180^\circ - P) : \sin(\sigma - \pi);$$

whence

$$E\,V_1 = E\,H \,.\, \frac{\sin(180^\circ - P)}{\sin(\sigma - \pi)} = E\,H \,.\, \frac{P}{\sigma - \pi} . \quad . \quad (147)$$

The least value for P is $52'\,50''$; the greatest value for $\sigma - \pi$ is $16'\,10''$; whence the length of the earth's shadow is always greater than three times the distance of the moon.

Fig. 86 bis.



§ 458. Again, in same triangle,

$$H E V_1 = E H A - E V_1 H;$$

and denoting the angle $H E V_1$ by E, we have

$$E = P + \pi - \sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)$$

The least value for $P + \pi - \sigma$ is $36' \, 41''$; whence the apparent semi-diameter of the section of the earth's shadow at the moon's distance is always greater than twice that of the moon; and this must be increased by about one-fiftieth of its value for the atmospheric absorption of the solar light which passes near the earth's surface. The moon may, therefore, enter completely within the earth's shadow.

§ 459. Denoting the angular distance $V_1 E M$ between the axis of the earth's shadow and moon's centre, at the beginning or ending of the lunar eclipse, by $\Delta_{\prime}$, we have

$$\Delta_{\prime} = E + s = P + \pi - \sigma + s \quad . \quad . \quad . \quad . \quad (149)$$

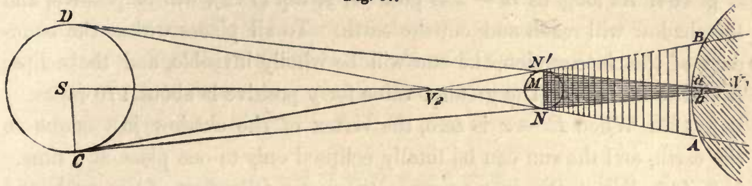
§ 460. The conical space on the opposite side of the earth from the sun, and of which the bounding surface is tangent to these bodies, and vertex between them, is called the earth's *penumbra*. Thus, $L B A L'$ is the earth's penumbra. Its apparent semi-diameter $L E V_1$, at the distance of the moon, denoted by $E_{\prime}$, is obtained from Eq. (148) by simply changing the sign of σ, these semi-diameters falling, in this case, on opposite sides of the axis $S E$; and we have

$$E_{\prime} = P + \pi + \sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150)$$

The moon experiences a loss of light from the instant she touches the penumbra, and this loss continues to increase till she enters the umbra or shadow.

§ 461. Now, let S be the sun, M the moon at new, E the centre of the earth, $B A$ an arc of the earth's surface—enlarged to avoid confusing the figure. The space $N V_1 N'$ is the moon's shadow, and $B N' N A$ her penumbra. To all places within the section of the former by the earth's surface, and of which $a b$ is the diameter, the sun will be totally, and to

Fig. 87.



all places within the annular space, of which $B\,a$ and $b\,A$ are sections, partially obscured, and present in the latter case a crescent phase. Make

$$r = ME = \text{moon's distance};$$
$$r_{,} = SE = \text{sun's distance};$$
$$d = MN = \text{moon's true semi-diameter};$$
$$d_{,} = SC = \text{sun's true semi-diameter};$$
$$x = E\,V_1 = \text{distance of conical vertex from earth's centre.}$$

Then, in the triangles $V_1\,S\,C$ and $V_1\,MN$, right-angled at C and N,

$$\frac{d}{r - x} = \frac{d_{,}}{r_{,} - x};$$

whence

$$x = \frac{d_{,}r - d\,r_{,}}{d_{,} - d};$$

and substituting the values of r, $r_{,}$, d, and $d_{,}$, as given by equations (28) and (29),

$$x = R\,.\,\omega\,.\,\frac{\sigma - s}{P\,\sigma - \pi\,s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

§ 462. Taking the values for P, π, σ, and s, which give this the greatest positive and negative values, it is found that the vertex V_1 sometimes falls short of the earth's centre about 7.6, and at others extends beyond that point about 3.5 times the earth's radius.

§ 463. Again, $R - x$ gives the distance of the vertex V_1, from the section of which $a\,b$ is the diameter. Denoting this latter by y, we have, in the triangles $a\,V\,b$ and $N\,V_1\,N'$,

$$r - x \;:\; R - x \;::\; 2d \;:\; y;$$

$$y = \frac{2\,d\,.\,(R - x)}{r - x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

and substituting the values of d, x, and r, from equations (29), (151), and (28), we find

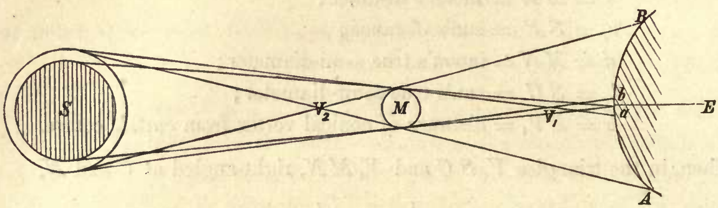
$$y = 2\,R\,.\,\frac{P\,\sigma - \pi\,s - \omega\,(\sigma - s)}{\omega\,.\,(P - \pi)} \quad . \quad . \quad . \quad . \quad (153)$$

§ 464. As long as $R - x$ is positive, y, Eq. (152), will be positive, and the shadow will reach and cut the earth. To all places within the boundary of this intersection, the sun will be wholly invisible, and the eclipse is said to be *total*. The greatest value for y positive is about 170 miles.

§ 465. When $R - x$ is zero, the vertex of the shadow just comes to the earth, and the sun can be totally eclipsed only to one place at a time.

§ 466. When $R - x$ is negative, the vertex falls short of the earth, and the surface of the latter intersects the opposite *nappe* of the conical shadow,

Fig. 88.



the value of y is, Eq. (152), negative; it measures the distance by which the opposite edges of the inner boundary of the penumbra overlap one another, and the space of which y negative is the diameter may be called the *umbral penumbra*, and is distinguished from the rest of the penumbra in embracing those points from which the sun appears as an unbroken *ring* around the black disk of the moon, while to all other points of the penumbra he will appear as *crescent*. In the first case the eclipse is said to be *annular;* in the second, *crescent*. The greatest possible diameter of the umbral penumbra is about 240 miles.

§ 467. To find AB, the diameter of the external boundary of the penumbra on the earth, it is only necessary to change the sign of s in Eq. (153), because in this case σ and s fall on opposite sides of the axis of the moon's shadow. Making this change in Eq. (153), we have

$$y_{,} = 2R \cdot \frac{P\sigma + \pi s - \omega \cdot (\sigma + s)}{\omega \cdot (P - \pi)} \quad . \quad . \quad . \quad . \quad (154)$$

The greatest value for which is about 4835 miles.

§ 468. The solar eclipse begins at the instant of first, and ends at the instant of last contact of the moon with the cone tangent to the sun and earth; and the places of first and last appearance on the earth are those at which the corresponding rectilinear elements of this cone are tangent to its surface. The solar eclipse being only visible to those places situated within the path of the penumbra, is, when considered with reference to the

whole earth, called *a general eclipse of the sun*, in contradistinction to its local character, of which more will be said presently.

§ 469. Let M_2 (Fig. 86) be the place of the moon at the beginning of a general eclipse of the sun; denote her angular distance $M_2 E S$ from the sun by $\varDelta$; then, since $EKB = P$, $E V_1 D = \sigma - \pi$, and $M_2 E K = s$, will

$$\varDelta = P + s + \sigma - \pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

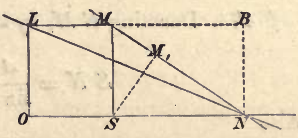
§ 470. In a lunar eclipse, if the moon's centre should cross the axis of the earth's shadow the eclipse is said to be *central.* When, during a solar eclipse, the centre of the sun, that of the moon, and the eye of the spectator are on the same right line, the eclipse is said to be *central.* If at time of syzygy the moon become tangent to the cone of the earth's shadow without entering, the phenomenon is called *appulse.*

§ 471. The atmospheric lens which envelops the earth causes the solar light passing through it to converge to a focus between the moon and earth, and this light diverging anew after concentration, and falling upon the lunar disk while in the earth's shadow, gives to it a dark, coppery-red illumination, and prevents total obscuration of the moon during a lunar eclipse.

Relative Geocentric Orbit of the Moon.

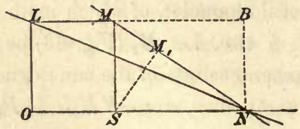
§ 472. The path which a body in motion appears to describe in reference to another also in motion, is called a *relative orbit;* and the distance of the one body from the other at any time will be the same whether we regard both as moving with their actual velocities, or one at rest and the other moving with a velocity of which the components in any three rectangular directions are equal to the differences of the components of the actual velocities in the same directions.

Fig. 89.



§ 473. Let $N O$ be an arc of the ecliptic, NL an arc of the lunar orbit projected upon the celestial sphere, N one of the nodes, S the point in which the axis of the earth's shadow pierces the celestial sphere when the moon is in her node, O the place of this point when the moon is either in conjunction or opposition at L. From the node to opposition or conjunction the moon will have described the arc NL and the axis of the earth's shadow, whose motion is always equal to the apparent motion of the sun, the arc $S\ O$. $L\ O$ is an arc of a circle of latitude; and assuming NL to represent the

Fig. 89 bis.



moon's actual velocity, $N O$ and $O L$ will be its components in longitude and latitude respectively. $S O$ is the velocity of the earth's shadow, which is wholly in longitude. If, therefore, we construct upon $N S = N O - S O$, and $S M = L O$, the rectangle $S B$, the diagonal $N M$ will be an arc of the relative orbit of the moon referred to the axis of the earth's shadow as an origin.

Ecliptic Limits.

§ 474. Hence, if $S M_1$ be drawn perpendicular to the relative orbit, the length of $S M_1$ will measure the nearest approach of the moon's centre to the axis of the earth's shadow. Make

m = moon's hourly motion in longitude;
n = sun's " " "
g = moon's hourly motion in latitude;
t = time from node to opposition or conjunction;
$\varphi = M N S$ = angle which the relative orbit makes with the ecliptic.

$$\tan \varphi = \frac{M S}{N S};$$

but supposing the motion uniform,

$$M S = g t;$$

also

$$N O = m t; \; S O = n t;$$

and

$$\therefore N S = m t - n t = (m - n) t;$$

whence

$$\tan \varphi = \frac{g}{m - n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (156)$$

The angle φ is then known.

§ 475. Denoting $S M_1$ by $\varDelta$, we have

$$S N = \frac{\varDelta}{\sin \varphi} = \varDelta \frac{\sqrt{(m - n)^2 + g^2}}{g} \quad . \quad . \quad . \quad . \quad (157)$$

Also

$$\frac{N O}{S N} = \frac{m}{m - n};$$

whence, substituting the value of $S N$,

$$N O = \varDelta \, . \, \frac{m}{m - n} \, . \, \frac{\sqrt{(m - n)^2 + g^2}}{g} \quad . \quad . \quad (158)$$

Making $\varDelta$ equal to that given in Eq. (149), the moon will just touch the earth's shadow, and $N\,O$ will become what is called the *ecliptic limit;* that is, *the least difference of longitude that can exist between the moon, and her nearest node at full*, to avoid an eclipse of the moon.

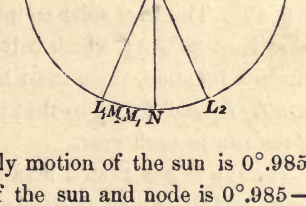
Taking the greatest value for $\varDelta$, $N\,O$ is found to be $12°\ 24'$, and least value it is found to be 9°. The first is called the greatest and the second the least *lunar ecliptic limit.* If therefore at the time of full moon the difference between the longitude of the moon and her nearest node exceed $12°\ 24'$, there cannot be an eclipse; if less than 9°, there must be one; if less than $12°\ 24'$ and greater than 9°, there may or may not, depending upon the inclination of the relative orbit and actual value of $\varDelta$. To solve the doubt, we have the given difference of longitude between $12°\ 24'$ and 9°, and the inclination φ, to find $S\,M_1$. If this latter be greater than $\varDelta$, there can be no eclipse; if less, there must be one.

§ 476. Again, making $\varDelta$ equal to that given in Eq. (155), and proceeding exactly as above, we find the greater and lesser *solar ecliptic limits.* The first is $18°\ 36'$ and the latter $15°\ 25'$.

Number of Eclipses.

§ 477. Let $N H N' H'$ be the ecliptic, N and N' the moon's nodes. Take $N L_1$, $N L_2$, $N' L_3$, and $N' L_4$, each equal to 18°.6, the greatest ecliptic limit. Then will $L_1\,L_2$ and $L_3\,L_4$ be each equal to 37°.2, and the number of new moons that can happen while the sun is apparently describing these arcs, will determine the number of solar eclipses that can occur in a single year.

Fig. 9C



The mean daily motion of the moon's node is $-0°.055$; the mean apparent daily motion of the sun is 0°.985, and hence the apparent relative motion of the sun and node is $0°.985 - (-0°.055) = 1°.04$. A lunation is 29.53 days; and $29.53 \times 1°.04 = 30°.7112$, say 30°.71, is the mean motion of the sun from the node in a lunation. Omitting the proper motion of the vernal equinox as insignificant in this estimate, its relative motion from the node in a lunation is $29.53 \text{ days} \times 0°.055 = 1°.6241$. These motions are incommensurable with each other and with 360°; and the vernal equinox, the node and sun with the moon in conjunction, will, in process of time, have any assumed positions with respect to each other at the beginning of the year.

Taking the sun at M_2, one degree to the east of L_1 (with moon in conjunction), will give one solar eclipse; and $37°.2 - 1° = 36°.2$ being greater than the arc described by the sun in a lunation, there will, at the end of the first lunation, be another solar eclipse between N and L_2. At the end of the sixth lunation, the sun will be at M_3, in advance of L_3 by a distance equal to $30°.71 \times 6 - 179° = 5°.26$, where there will be a third solar eclipse; and $37°.2 - 5°.26 = 31°.84$ being greater than arc described in a lunation, there must be a fourth solar eclipse before the sun passes L_4. At the end of the twelfth lunation, the sun will be $30°.71 \times 12 - 360° = 8°.52$ to the east of M_2, his initial place, where there will be a fifth solar eclipse, and this will be the last within the year, which will end 10.89 days after, this being the excess of the year over twelve lunations.

Again, $18°.6 - 1° = 17°.6$; and as in a semi-lunation the sun will pass over $15°.35$ of this arc, he will come to the distance $17°.6 - 15°.35 = 2°.25$ from the node N_3, and there will be a first lunar eclipse at the opposite node N_1. When the moon was new at M_3, the sun was $18°.60 - 5°.36 = 13°.34$ from the node N, and in half a lunation after will be $15°.35 - 13°.34 = 2°.01$ beyond it, and there will be a second lunar eclipse, and no more within the year, for the next lunation will carry the sun beyond the lunar ecliptic limits. Had the initial place of the sun been taken at M_1, at a distance $4°.26$ to the west of the node N, at the beginning of the year, and the moon in opposition, it might have been shown that there would have been four eclipses of the sun and three of the moon.

§ 478. The least solar ecliptic limit being $15°.42$, the arc $L_1 L_2$ must be at least $30°.84$; which being greater than the arc passed over by the sun in a lunation, there must be at least one solar eclipse in each of the arcs $L_1 L_2$ and $L_3 L_4$, so that there must always be at least two eclipses of the sun in each year.

The sun is less than a lunation in passing through the lunar ecliptic limits, and there may, therefore, be no eclipse of the moon within the year.

To sum up, then, there may be seven eclipses within the year, and there may be only two. In the former case, five may be of the sun and two of the moon, or four of the sun and three of the moon; and in the latter, both must be of the sun.

The Saros.

§ 479. The synodic period of the moon is 29.53058, and that of the moon's node 346.6196 days. These numbers are to one another as 19 to 223 nearly. If, therefore, the moon and her nodes be in syzygy at the

Plate VI.

TO FRONT PAGE 127.

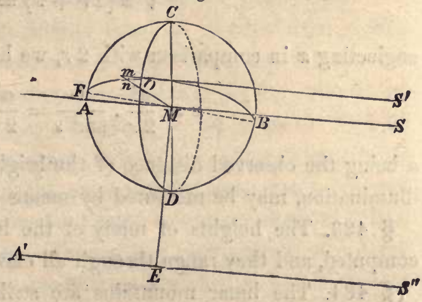
same time, they will be so again after 19 revolutions of the node, or 223 lunations; so that the eclipses will recur again very nearly in the same order within the same period, which is about 18.027 years. This period is known as the Chaldean *Saros*. There are generally 70 eclipses in the saros, of which 29 are lunar and 41 solar.

PHYSICAL CONSTITUTION OF THE MOON.

§ 480. Telescopes disclose certain varieties of illumination on the moon's surface, which can only arise from mountains and valleys. The shadows cast by the former lie in directions and are of lengths required by the inclination of the solar rays to that portion of the moon's surface on which the mountains stand. The convex outline of the moon turned towards the sun is always circular and nearly smooth; but the opposite or elliptical border of the illuminated part is extremely ragged, and indented with deep recesses and prominent points. To places along this line the sun is just rising, and the neighboring mountains cast long black shadows on the plains below. As the sun rises these shadows shorten; and at full moon, when the solar light penetrates the mountain valleys and shines on every point of the field of view, no shadows are seen.

§ 481. The summits of the lunar mountains often appear as small bright points, or islands of light, beyond the edge of the illuminated part, as they catch the sunbeams before the intervening plains. As the sun advances in altitude, these luminous patches expand, and finally unite with the general illumination, and the mountains appear as projections from its elliptical border.

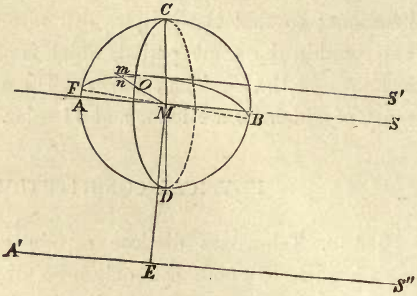
Fig. 91.



§ 482. To compute the height of a lunar mountain, let E, M, and S be the centres of the earth, moon, and sun respectively; $A\,C\,B\,D$ and $D\,O\,C$ sections of the general surface of the moon by planes respectively perpendicular to EM and MS; then will the visible illuminated part of the disk be contained between $C\,B\,D$ and the projection of $D\,O\,C$ on the section $A\,C\,B\,D$. Also let m be the top of a mountain just catching the solar rays that graze the general surface of the moon at O; $B\,O\,F$ the arc of a

great circle of this surface, and of which the plane passes through the top of the mountain and centres of the sun and moon; n the point in which this arc is cut by the line Mm drawn from the top of the mountain to the moon's centre.

Fig. 91 bis.



Make

$r = Mn$ = radius of the moon;
$\varepsilon = FCO = EMA$ = exterior angle of elongation;
$y = Om$ = distance of m from O;
$x = mn$ = height of mountain;
a = the projection of y on the plane $ACBD$.

Then, since the ray $S'Om$ is perpendicular to the section DOC, it is inclined to the section $ACBD$, under an angle equal to the complement of $FCO = 90^\circ - \varepsilon$; and we have

$$a = y \cdot \cos(90^\circ - \varepsilon) = y \cdot \sin \varepsilon;$$

whence

$$y = \frac{a}{\sin \varepsilon}.$$

Also

$$y = \sqrt{x(2r + x)};$$

and, therefore,

$$\sqrt{x(2r + x)} = \frac{a}{\sin \varepsilon};$$

neglecting x in comparison with $2r$, we have

$$x = \frac{a^2}{2r} \cdot \frac{1}{\sin^2 \varepsilon} = \frac{a^2}{2r} \cdot \operatorname{cosec}^2 \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

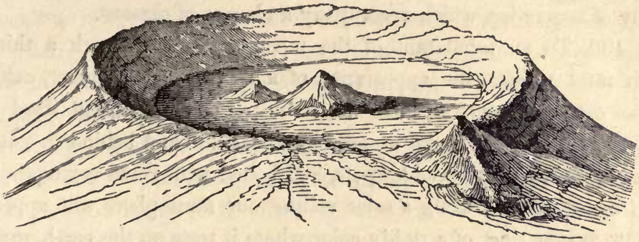
a being the observed distance of the bright spot from the boundary of the illumination, may be measured by means of the micrometer.

§ 483. The heights of many of the lunar mountains have been thus computed, and they range through all elevations up to 23,000 English feet.

§ 484. The lunar mountains are strikingly uniform in aspect. They are very numerous, especially towards the southern border, occupying by far the larger portion of the surface. They present almost universally a circular or cup-shaped form in ground plan, which becomes foreshortened into an ellipse towards the limb. The larger of these cups have for the most

part flat bottoms, from each of which rises centrally a small, steep, conical hill, presenting in all respects the true volcanic character as exhibited by

Fig. 92.



like districts on the earth, but with this peculiarity, viz.: that the bottoms are so deep as to lie below the general surface of the moon, the internal depth being often twice or thrice the external height.

§ 485. The heights of mountains in the immediate vicinity of each other being proportional to the length of their respective shadows, the depths of the pits or craters are easily computed from the heights of the edges above the general level, and the lengths of the shadows they cast internally and externally.

§ 486. Through the Rosse telescope, the flat bottom of the crater called *Albategnius*, is seen to be strewed with blocks not visible through inferior instruments; and the exterior of another, called *Aristillus*, is hatched over with deep gullies, radiating from a centre.

§ 487. There are also extensive tracts of the lunar surface which are perfectly level, and present decided indications of an alluvial character, and yet there is a total absence of all appearances of deep water.

§ 488. There are no clouds, or other indications of an atmosphere.

A lunar atmosphere of a mean density equal to 1980th that of the earth, would give a horizontal refraction of 1″, and cause the diameter of the moon, measured with a micrometer and estimated by the interval of a star's disappearance in an occultation, to differ; would cause the limb of the moon, during a solar eclipse, to appear beyond the cusps externally to the sun's disk as a narrow line of light, extending for some distance along the edge; and would extinguish very faint stars before occultations. But none of these phenomena are seen. During the continuance of a total lunar eclipse, when the light of the moon is so deadened as not to obliterate by contrast the feeble light of the smaller stars, the latter are seen to come up to the moon's limb and undergo sudden extinction, without any apparent displacement.

§ 489. The light from the moon developes but feeble heat, for even

when collected into the foci of large reflectors, it affects but little the thermometer; and there are no appearances indicating the slightest change of surface, such as would result from the periodical growth and decay of vegetation which accompany a change of seasons.

§ 490. To an inhabitant of the moon, if there be such a thing, the earth must present the appearance of a moon 2° in diameter, exhibiting phases complementary to those the moon presents to us, but fixed in the sky, while the stars seem to pass slowly beside and behind it. It must appear clouded with variable spots, and belted with zones corresponding to our trade-winds. During a solar eclipse our atmosphere will appear as a narrow, bright ring, of a ruddy color where it rests on the earth, gradually passing into faint blue, encircling the whole or part of the earth's disk.

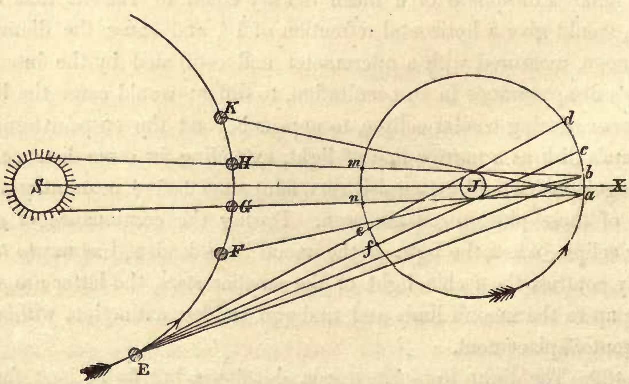
SATELLITES OF JUPITER.

§ 491. The satellites of Jupiter, four in number, revolve about their primary from west to east in planes nearly coincident with that of the planet's equator, and but slightly inclined to the ecliptic.

§ 492. Their orbits appear, therefore, projected very nearly into straight lines, in which they oscillate to and fro, sometimes passing between the sun and Jupiter, causing an eclipse of the sun to the latter, sometimes entering the planet's shadow and being themselves eclipsed, and sometimes disappearing either behind the body of Jupiter or in transiting his disk.

§ 493. Thus, let S be the sun; E, the earth, of which the orbit is $EFGH$; J, Jupiter; and $efab$, the orbit of a satellite. The cone of Jupiter's shadow will have its vertex at X, far beyond the orbit of the satel-

Fig. 93.



lite, and the penumbra, owing to the great distance of the sun and consequent smallness of the angle at Jupiter subtended by his disk, will extend but little beyond the shadow within the limits of the satellite's orbit. The satellite revolving from west to east, will cast a shadow upon Jupiter while passing from *m* to *n*, will transit his disk from *e* to *f*, enter his shadow at *a*, emerge from it at *b*, and disappear behind the body of the planet while passing from *c* to *d*.

§ 494. The shadows of the satellites are frequently seen crossing the disk of Jupiter. While in the act of transiting, the satellite generally disappears, its light being confounded with that of the planet, unless it happens to be projected upon a dark belt, in which case it is visible. Under these circumstances it occasionally appears as a dark spot smaller than its shadow, which has led to the conclusion that certain of the satellites have now and then on their own bodies, or within their atmospheres, obscure spots of great extent.

§ 495. From the eclipses of the satellites are obtained all the data for the determination of the laws of their motions. These eclipses are in general analogous to those of the moon, but in their details they differ considerably. The great distance of Jupiter from the sun and his great size, make his shadow much larger and longer than that of the earth. The satellites are much smaller in proportion to their primary, and their orbits less inclined to his ecliptic, than in the case of the moon. From these causes the three interior satellites enter the shadow at every revolution, and are totally eclipsed; and although the fourth, from the greater inclination and distance of its orbit, sometimes escapes eclipse, yet it does so seldom.

§ 496. Besides, these eclipses are not seen by us from the centre of motion, as are those of the moon, but from some remote station, of which the place with respect to the shadow is ever changing. And while this circumstance makes no difference in the *time* of the eclipses, it yet affects materially the *visibility* and the *apparent relative situations* of the planet and satellites at the instant of the latter's entering and quitting the shadow

§ 497. A satellite never enters the shadow suddenly because of its sensible diameter, and the time from the first perceptible loss of light to its total extinction will be that required by the satellite to describe about Jupiter an angle equal to its apparent diameter as seen from the planet's centre. The same is true of the emergence. Owing to the difference in telescopes and eyes, this becomes a source of discrepancy in the times assigned by different observers for the beginning and ending of an eclipse. But if both the immersion and emersion be observed by the same person and with the same telescope, the half sum of the two times, as given by a properly

regulated time-keeper, will be that of apparent opposition measurably free from error.

§ 498. The intervals between the oppositions give the synodic period, which, in Eq. (146), will give the mean motion, knowing that of Jupiter, and hence the sidereal period. Eq. (142).

The satellites are named *first*, *second*, *third*, and *fourth*, according to their order of distance from Jupiter.

The elements of the satellites' orbits will be found in the following

Table.

Sat.	Sidereal period.	Mean distance. Rad. of J=1.	Inclination of orbit to a fixed plane proper to each	Inclination of the fixed plane to Jupiter's equator.	Retrograde revolution of nodes on fixed plane.	Mass: that of Jupiter 1,000,000,000.
	d. h. m. s.		° ′ ″	° ′ ″	Years.	
1st.	1 18 27 33.506	6.04853	0 0 0	0 0 6		17328
2d.	3 13 14 36.393	9.62347	0 27 50	0 1 5	29.9142	23235
3d.	7 03 42 33.362	15.35024	0 12 20	0 5 2	141.7390	88497
4th.	16 16 31 49.702	26.99835	0 14 58	0 24 4	531.0000	42659

It will assist in forming some idea of the relative dimensions of Jupiter and his satellites to examine the following

Table.

	Mean apparent diameter as seen from earth.	Mean apparent diameter as seen from Jupiter.	Diameter in miles.	Mass.
	″	′ ″		
Jupiter.	38.327		87000	1.0000000
1st sat.	1.017	33 11	2508	0.0000173
2d "	0.911	17 35	2068	0.0000232
3d "	1.488	18 00	3377	0.0000885
4th "	1.273	8 46	2890	0.0000427

From which it follows that the first satellite appears to a spectator on Jupiter as large as our moon to us; the second and third nearly equal to each other, and somewhat more than half the size of the first; and the fourth about a quarter of that size. They frequently eclipse each other. The apparent diameters of the planet as seen from the satellites are 19° 49′; 12° 29′; 7° 47′; 4° 25′.

§ 499. Figure 93 shows that the eclipses take place to the west of Jupiter, while the latter is moving from conjunction to opposition, and to the

east from opposition to conjunction. As Jupiter approaches to opposition, the line of sight from the earth becomes more nearly coincident with the direction of the shadow, and the place of the eclipse will be nearer and nearer to the body of the planet. When the earth comes to F, from which a line drawn tangent to the body of the planet will pass through b, the emersion will cease to be visible, and will, up to the time of opposition, take place behind the planet. Similarly, from opposition up to the time when the earth arrives at K, the immersion will be concealed from view. These remarks apply particularly to the third and fourth satellites, the proximity of the others to the planet being so great as to make it impossible ever to see the immersion and emersion both at the same eclipse.

§ 500. The mean motions of the satellites are connected by this remarkable law, viz: If the mean angular velocity of the first satellite be added to twice that of the third, the sum will equal three times that of the second. If, therefore, from the mean longitude of the first satellite, increased by twice that of the third, three times the mean longitude of the second be subtracted, the remainder will be a constant quantity and this constant is found to be equal to 180°. This Laplace has shown to be a consequence of the mutual attractions of the satellites for one another. The first three satellites cannot, therefore, be eclipsed at the same time.

§ 501. While, however, the satellites cannot all be eclipsed at once, they may be, and, indeed, occasionally are, all invisible by the simultaneous eclipse of some, occultations of others, and transits of the rest.

§ 502. The orbits of the satellites are but slightly eccentric, the two inferior ones not at all so, so far as observation is capable of revealing eccentricity. Their mutual attractions produce in them perturbations analogous to those of the planets about the sun. These are investigated in physical astronomy.

§ 503. By careful observations the satellites are found to exhibit marked fluctuations in respect to brightness. These fluctuations happen periodically, and appear connected with the position of the satellites with respect to the sun; from which it is inferred that they revolve upon their axes like our moon, each once in its sidereal period.

§ 504. At one time the eclipses of Jupiter's satellites were much used in the determination of terrestrial longitude, but more modern methods, free from the objections referred to in § 497, have in a measure supplanted them.

Progressive Motion of Light.

§ 505. To these eclipses science is indebted for the discovery of the successive propagation and velocity of light.

The earth's orbit being concentric with that of Jupiter and interior to it, the distance of these bodies is continually varying, the variation extending from the sum to the difference of the radii of the two orbits, making the excess of the greatest over the least distance equal to the diameter of the earth's orbit. Now, it was observed by Roemer, a Danish astronomer, on comparing together the eclipses during many successive years, that those which took place about opposition were observed earlier, and those about conjunction later than an average or mean time of occurrence. And connecting the observed acceleration in the one case and retardation in the other with the variation of Jupiter's distance below and above its average value, he found the difference fully and accurately accounted for by allowing $16^m\ 26^s.6$ for light to traverse the diameter of the earth's orbit. In other words, using the figure of a cord moving in the direction of its length from the satellite to the earth to illustrate the flow of luminous waves in the same direction, if the cord were severed at the edge of Jupiter's shadow, the severed end would be $16^m\ 26^s.6$ longer in reaching the earth when the planet is in conjunction than in opposition, having a greater distance to travel in the first case by the diameter of the earth's orbit = 190,000,000 miles, than in the second. The satellite is seen long after it has entered the shadow, and is invisible long after it has emerged from it. Dividing the diameter of the earth's orbit by $16^m\ 26^s.6$ reduced to seconds, the velocity of light is found to be 192,000 miles a second.

SATELLITES OF SATURN.

§ 506. Eight satellites are known to accompany Saturn. They revolve about him from west to east, and in planes nearly coincident with that of the planet's ring, except the eighth, whose orbit is inclined to this latter plane under an angle of about $12° 14'$. This satellite is also distinguished from the others by its remoteness from the planet, its distance being 2.3 times that of the most distant of the others, and equal to 64 times the equatorial radius of Saturn, resembling in this respect our own moon. It is also remarkable for the exhibition of greater variety of illumination in different parts of its orbit than any other known secondary. Indeed, so feeble is the light which it reflects to the earth when to the east of Saturn that it becomes invisible through ordinary telescopes; and from this defi-

ciency of light occurring constantly on the same side of Saturn, *as seen from the earth*, it is inferred that this satellite revolves on its axis once during its sidereal period.

§ 507. The next in order, proceeding inwardly, is so obscure as to have eluded the observations of astronomers until very recently. It was discovered simultaneously by Mr. Bond, of Cambridge, U. S., and Mr. Lassell, of Liverpool, England, in 1848.

§ 508. The next in order, proceeding in the same direction, is by far the largest and most conspicuous of all, and probably not inferior to Mars in size.

§ 509. The next three in order are very small, and require pretty powerful telescopes to see them, while the two interior, which just skirt the edge of the ring, can only be seen with telescopes of extraordinary power and perfection, and under the most favorable atmospheric circumstances. When first discovered, they appeared to thread the excessively thin film of light reflected from the edge of the ring then turned towards the earth, and for a short time to advance off at either end, speedily to return again.

§ 510. Owing to the obliquity of their orbits to the plane of Saturn's ecliptic, there are no eclipses, occultations, or transits of the satellites, or shadows on the disk of the primary, except at the time when the ring is seen edgewise, and their observation is attended with too much difficulty to be of any practical use, like the corresponding phenomena of Jupiter's satellites, for the determination of terrestrial longitude.

§ 511. The names and elements of Saturn's satellites are given in the following

Table.

Names and Order of Satellites.	Sidereal Period.	Mean Distance.	Epoch of Elements.	Mean Longitude at the Epoch.	Eccentricity.	Perisaturnum.
	d. h. m. s.			° ′ ″		
1. Mimas ...	0 22 37 22.9	3.3607	1790.0	256 58 48		
2. Enceladus	1 08 53 06.7	4.3125	1836.0	67 41 36		
3. Tethys...	1 21 18 25.7	5.3396	"	313 43 48	0.04 ?	54° ?
4. Dione....	2 17 41 08.9	6.8398	"	327 40 48	0.02 ?	42 ?
5. Rhea	4 12 25 10.8	9.5528	"	353 44 00	0.02 ?	95 ?
6. Titan	15 22 41 25.2	22.1450	1830.0	137 21 24	0.029314	256° 38′.11
7. Hyperion.	22 12 ? ?	28. ±				
8. Iapetus ..	79 07 53 40.4	64.3590	1790.0	269 37 48		

The longitudes are reckoned in the plane of the ring from its descending node on the ecliptic. The apsides of Titan have a direct motion of 30′ 28″ per annum in longitude on the ecliptic.

§ 512. The periodic times of the first four satellites in order of distance from Saturn are connected by this law, viz.: The period of the third is double that of the first, and the period of the fourth is double that of the second; the coincidence being exact to within $\frac{1}{800}$ part of the larger period.

SATELLITES OF URANUS.

§ 513. Uranus is believed to have six satellites, which revolve about the primary from east to west, in orbits nearly, if not quite, circular, and which make with the ecliptic an angle of 78° 58′. They thus differ from all the other known bodies of the solar system both in the direction of their motion and inclination of their orbits, which latter, as well as the places of the nodes, have undergone no sensible change, during at least one-half of the planet's period around the sun.

The elements of these satellites, as far as known, are given in this

Table.

Sat.	Sidereal Revolution.	Mean Distance.	Epoch of passing Ascending Node.	Nodes and Inclination.
			Gr. T.	
1	4^d ? ^{h}			Inclination of orbits to the ecliptic, 78° 58′; ascending node in longitude, 165° 30′. (Equinox of 1798.) Motion *retrograde*, and orbits nearly circular.
2	8 16^h 56^m $31^s.3$	17 0	1787. Feb. 16, 0^h 10^m	
3	10 23 ?	19 8?		
4	13 11 07 12.6	22 8	1787. Jan. 7, 0^h 28^m	
5	38 2 ?	45 5?		
6	107 12 ?	91 0?		

§ 514. The satellites of Uranus require very powerful and perfect telescopes for their observation. The second and fourth are far the most conspicuous, and their periods and distance have been ascertained with tolerable certainty. The first and third have also been observed since their original announcement, but of the existence of the fifth and sixth we have not the same evidence. Sir John Herschel is of opinion that if future observations should assign them places, they would be exterior to that of the fourth.

515. When the earth is in the plane of the orbits or nearly so, the apparent paths of the satellites are straight lines or very elongated ellipses, in which case these secondaries become invisible long before they come up to the disk of the planet, in consequence of the superior light of the latter, so that it is not possible to observe their occultations, eclipses, and transits.

SATELLITES OF NEPTUNE.

§ 516. If the observation of the satellites of Uranus be difficult, those of Neptune, owing to the great distance of this planet, must offer still greater difficulties. Of the existence of one satellite there remains no doubt. Its sidereal period about the planet is nearly 5.9 days; its mean distance is fourteen times Neptune's semi-diameter; and its orbit is inclined to the plane of the ecliptic under an angle of about 35°.

COMETS.

§ 517. Comets differ from all the primary bodies with which we have thus far been concerned, in their appearance, the shape and inclination of their orbits, and in following no rule, as a class, with regard to the direction of their motions. They are of various sizes, some being visible to the naked eye even in daytime, while others require the aid of telescopes even at night to see them.

§ 518. The larger consist for the most part of an ill-defined mass, called the *head*, from which, *in a direction opposite the sun*, proceeds a train, of greater or less extent, called the *tail.*

Fig. 94.

§ 519. The head is much brighter towards its centre. Sometimes this increase of illumination terminates in a bright spot, called a *nucleus*, the surrounding haze which makes up the rest of the head being called the *coma*.

§ 520. The tail appears to consist of two streams of luminous matter which, starting from a point near the head, and on the side towards the sun, pass suddenly to the opposite side, and grow broader and more diffused as they increase in length; they commonly unite at a little distance from the head, but sometimes continue distinct for the greater part of their course. This appendage has been known to attain the enormous length of forty-one millions of miles, and to stretch over 104 degrees of the celestial sphere.

§ 521. The tail is not, however, an invariable appendage of comets,

many of the brightest having been seen with little or none, and others as round and well-defined as Jupiter.

§ 522. On the other hand, there are instances of comets with many tails or streamers, spreading out like an immense fan, and extending to the distance of some 30 degrees of the celestial vault. One is recorded as having two tails, making with each other an angle of 160°, the fainter being turned towards, the other from the sun.

The tails are often curved, bending, in general, towards that part of space which the comet has left, as if retarded by the opposition of some resisting medium.

§ 523. The smaller comets, such as are only visible through telescopes, and which are by far the most numerous, present no appearance of a tail, and seem as round or oval vaporous masses, more luminous towards the centre, where, in some instances, a small stellar point has been seen, but without any distinct nucleus or other signs of a solid body. Stars of the smallest magnitude, such as would be obliterated by a moderate fog, are seen through their brightest part.

§ 524. A comet never exhibits the least signs of phases; but, on the contrary appears as a mass of thin vapor, either self-luminous, or easily penetrated by the luminous waves from the sun, which are reflected from its interior parts as from its exterior surface.

§ 525. The tail, where it comes up and surrounds the head, is yet separate from the latter by an interval less luminous, as if sustained and kept from contact by a transparent stratum of atmosphere; and seems to be a kind of hollow envelope of a parabolic form, inclosing the head near its vertex.

§ 526. The number of recorded comets is very great, amounting to several hundred; and when it is considered that in the earlier stages of astronomy, before the invention of the telescope, only large and conspicuous ones could be noticed, and that, since due attention has been paid to the subject, scarcely a year passes without the observation of one or two of these bodies, and sometimes two or three have appeared at once, it may very reasonably be supposed that many thousands exist Multitudes must escape observation by reason of their paths traversing only that part of the heavens which is above the horizon in daytime. Comets so circumstanced can only become visible during a total eclipse of the sun—a coincidence which is related to have taken place sixty years before Christ, when a a large comet was observed near the sun.

§ 527. The motion of comets is characterized by the greatest irregularity. Sometimes they appear in sight for a few days only, at others for

many months. Some move very slowly, others with vast velocity; and not unfrequently the two extremes of speed are exhibited by the same individual in different parts of its path. Some pursue a direct, others a retrograde, and others a tortuous and very irregular course; nor are they confined, like the planets, to any particular region of the heavens, but traverse indifferently every part alike.

§ 528. Their variations in apparent size, while visible, are equally remarkable; sometimes they make their appearance as faint, slow-moving objects, with little or no tail; by degrees they accelerate their speed, enlarge and extend their tail, which increases in length and brightness till they approach the sun near enough to be lost in his light. After a time they again emerge on the opposite side, receding from the sun. It is now for the most part they shine forth in all their splendor, and display their tails in greatest length and development. As they continue to recede from the sun their motion diminishes, their tails subside about the head, which grows continually feebler till lost in the distance, from which by far the greater number have never returned; thus indicating their paths to be along the parabola or hyperbola.

§ 529. These seemingly irregular and capricious movements are fully explained by the doctrine of universal gravitation, and are no other than consequences of the laws of elliptic, parabolic, or hyperbolic motions. But the physical changes of the head, the process by which it builds up the enormous tail, takes it down again, and wraps it as a mantle about itself; the position of the tail as regards the direction of the sun, the multiplicity of tails, and other physical phenomena to be noticed presently, remain without satisfactory solution.

§ 530. The elements of a comet's orbit are readily computed from *three* observed places, exactly as in the case of a planet; and the comet usually takes the name of the computor who thus first defines its track through the heavens.

The elements of a few now reckoned among the permanent members of the solar system, will be found in the following table:

TABLE.

ELEMENTS OF THE ORBITS OF THE PERMANENT COMETS.

COMET.	τ	π	Ω	ι	a	ϵ	P	DIRECTION.
	d. h. m. s.	° ′ ″	° ′ ″	° ′ ″			y.	
Halley	1835 Nov. 15 22 41 22	304 31 32	55 09 59	17 45 05	17.98796	0.967391	76.30	Retrograde.
Encke...........	1845 Aug. 9 15 11 11	157 44 21	334 19 33	13 07 34	2.21640	0.847436	3.30	Direct.
Biela............	1846 Feb. 11 0 02 50	109 05 47	245 56 58	12 34 14	3.50182	0.755471	6.55	do.
Fay.............	1843 Oct. 17 3 42 16	49 34 19	209 29 19	11 22 31	3.81179	0.555962	7.44	do.
De Vico.........	1844 Sept. 2 11 36 53	342 31 15	63 49 31	2 54 45	3.09946	0.617256	5.45	do.
Brorson..........	1846 Feb. 25 9 13 35	116 28 34	102 39 36	30 55 07	3.15021	0.793629	5.59	do.

In which

$\tau =$ time of perihelion passage;

$\pi =$ longitude of perihelion;

$\Omega =$ longitude of ascending node for epoch of perihelion;

$\iota =$ inclination to the ecliptic;

$a =$ semi-major axis;

$\epsilon =$ eccentricity;

$P =$ periodic time in years.

§ 531. By far the most interesting of these comets is that of Halley. Its last return took place according to prediction in 1835. While yet remote from the sun in its approach to that luminary, its appearance was that of an oval nebula without tail, and having a minute point of concentrated light eccentrically situated within. Soon its tail began to be developed, and increased rapidly till it reached its greatest length, about 20 degrees, when it decreased with such haste as to disappear entirely before perihelion passage. When the .ail first began to form, the nucleus became much brighter, and threw out a jet or stream of light towards the sun. This ejection continued, with occasional intermission, as long as the tail continued visible. Both the form and direction of this luminous stream underwent singular and capricious alterations, the different phases succeeding one another with such rapidity that no two successive nights presented the same appearance. At one time the jet was single, at others fan-shaped, while at others two, three, or more jets were darted forth in different directions, the principal one oscillating to and fro on either side of the line drawn to the sun. These jets, though very bright at their point of emanation from the nucleus, faded away, and became diffused as they expanded into the coma, at the same time curving backward as if thrown against a resisting medium. After its perihelion passage, the comet was not seen for two months, and at its reappearance presented itself under a new aspect. There was no longer a vestige of tail; it seemed to the naked eye a hazy star of the fourth magnitude, and through a powerful telescope a small round well-defined disk, rather more than 2′ in diameter, surrounded by a nebulous coma of much greater extent. Within the disk, and somewhat removed from its centre, appeared a minute but bright nucleus, from which extended, in a direction opposite the sun, a short vivid luminous ray. As the comet receded from the sun, the coma disappeared, as if absorbed into the disk, which increased so rapidly as in one week to augment its volume in the ratio of 40 to 1. And so it continued to swell out, with undiminished rate, until from this cause alone it ceased to be visible, the illumination becoming fainter as the magnitude increased. While this increase of dimensions proceeded, the form of the disk passed, by gradual and successive additions to its length in the direction opposite to the sun, to that of a paraboloid, the side towards the sun preserving its planetary sharpness, but the base being so faint and ill-defined, as to indicate that if the process had been continued with sufficient light to render it visible, a tail would ultimately have been observed. The parabolic envelope finally disappeared, and the comet took its leave as it came—a small round nebula, with a bright point in or near the

centre. Figures 5 to 10 inclusive, of plate, taken in order, show some of the successive aspects of this comet at its last appearance.

§ 532. Many other great comets are recorded, all affording peculiarities more or less interesting.

§ 533. On comparing the intervals between the successive returns of Encke's comet, its periods are found to be continually shortening; that is, its mean distance from the sun, or semi-major axis of its orbit, diminishes by slow and regular degrees, and at the rate of about $0^d.11$ during each revolution. This is attributed to the resistance of the ethereal medium which fills the planetary space, and serves as the medium for the transmission of light. This resistance checks the velocity, diminishes the centrifugal force, and gives to the sun more effect in drawing the comet towards itself. It will probably ultimately fall into that body. Like the comet of Halley, its apparent diameter is found to diminish as it approaches to, and to increase as it recedes from the sun. It has no tail, and presents to the view only a small ill-defined nucleus, eccentrically situated within a more or less elongated oval mass of vapors, being nearest to that vertex which is towards the sun.

§ 534. Biela's comet is scarcely visible to the naked eye; its orbit nearly intersects that of the earth, and had the latter, at the time of its passage in 1832, been a month in advance of its actual place, it would have passed through the comet.

At its last appearance it separated itself into two parts, which continued to journey along together, side by side, through an arc of 70 degrees of their orbit, keeping all the while within the same field of view of a telescope directed towards them. Both had nuclei, both had short tails parallel to one another, and perpendicular to their line of junction. At first the new comet was extremely small and faint in comparison with the old: the difference both in light and size diminished till they became equal; after which the new comet gained the superiority of light, presenting, according to Lieut. Maury, the appearance of a diamond spark. The old comet soon, however, recovered its superiority, and the new one began to fade, till finally the comet was seen single before it disappeared. While this interchange of light was going on, the new comet threw out a faint bridge-like arch of light, which extended from one to the other. When the original comet recovered its superior brightness, it in its turn threw forth additional rays, so as to present the appearance of a comet with three tails, forming with one another angles of about 120°. The distance between the comets at one time was about 39 times the equatorial radius of the earth, or less than two-thirds the distance of the moon from the earth.

Plate VII.

FIG. 5.

FIG. 6.

FIG. 7.

FIG. 8.

FIG. 9.

FIG. 10.

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§ 535. The orbits of comets being very eccentric, and inclined under all sorts of angles to the ecliptic, these bodies must pass near to the planets, and be more or less affected by their disturbing action.

One passed Jupiter at the distance of $\frac{1}{58}$ of the radius of that planet's orbit, and the earth, three years afterwards, at seven times the moon's distance. This comet was found by Lexell to have passed its perihelion in an elliptical orbit, of which the eccentricity was 0.7858, and with a periodic time of about five and a half years, having, in all probability, been drawn into this path by the perturbating action of Jupiter and the earth at its previous visits. Its next return could not be observed by reason of the relative places of its perihelion and of the earth, and before another revolution could be accomplished, it passed within the orbit of Jupiter's fourth satellite, and has never been seen since. The action of Jupiter doubtless changed its orbit into an extremely elongated ellipse, or perchance into a parabola or hyperbola; and what is most remarkable, none of Jupiter's satellites suffered any perceptible derangement—a sufficient proof of the smallness of the comet's mass.

§ 536. The great number of comets which appear to move in parabolic orbits, or elliptical orbits so elongated as not to be distinguished from them, has given rise to an impression that these bodies are extraneous to our system, and that our elliptic comets owe their permanent denizenship within the sphere of the sun's dominant attraction to the retarding action of one or other of the planets near which they may have passed, and by which their velocity was reduced to compatibility with elliptic motion. A similar disturbing cause, acting to increase the velocity, would give rise to a parabolic or hyperbolic orbit, so that it is not impossible for a comet to be drawn into our system, retained during many revolutions about the sun, and finally expelled from it, never more to return, as was probably the case with that of Lexell.

§ 537. The fact that all the planets and nearly all the satellites move in one direction about the sun, while retrograde comets are very common, would go far to assign them an extraneous origin. From a consideration of all the cometary orbits known in the early part of the present century, Laplace found that the average situation of their planes was so nearly perpendicular to the ecliptic as to afford no presumption of any cause biasing their inclinations. And yet as the planes of the elliptical orbits approach that of the ecliptic, the number of direct comets increases; and a plane of motion coincident with that of the earth, and periodicity of return, are decidedly favorable to direct motion.

STARS.

§ 538. Besides the bodies composing the solar system, there are a countless multitude of others which, because they retain their relative places sensibly unchanged are called, though improperly, *fixed stars.* Like our sun they are poised in space, are self-luminous, and in all probability are centres of planetary systems.

§ 539. Among these stars, which at first view seem scattered over the celestial vault at random, appears, every evening, a bright band, called the *milky way*, that stretches from horizon to horizon and forms a zone completely encircling the heavens. It divides in one part of its course into two branches, which unite again after remaining separate for 150° of their course.

§ 540. The most refined observations have been able to assign to none of the stars a sensible *geocentric*, and to but very few only an exceeding small and uncertain *annual parallax;* while the most powerful magnifiers have thus far failed to reveal an appreciable disk.

But little can, therefore, be known of their distances, nothing at all of their real dimensions, and the only means by which one may be distinguished from another are in the character and intensity of their illumination.

§ 541. It is usual to arrange the stars into classes called *magnitudes*, and this without reference to their location in the heavens. The brightest are said to be of the *first magnitude*, those which fall so far short of the first degree of brightness as to make a strongly marked distinction, are classed in the *second*, and so on down to the *sixth* or *seventh*, which comprise the smallest stars visible to the naked eye in the clearest and darkest night.

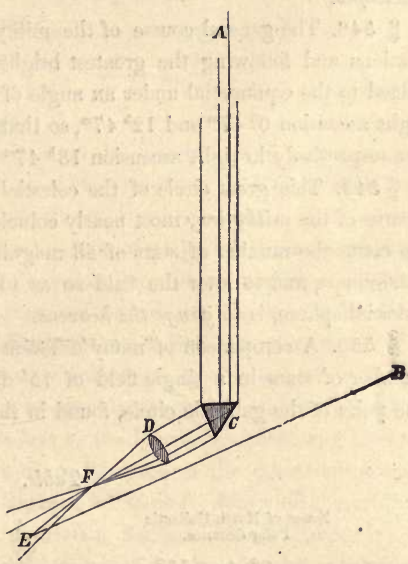
§ 542. Beyond this, however, telescopes continue the range of visibility down to the 16th; nor does there seem any reason to assign a limit to the progression, for every increase in the dimensions and power of telescopes has brought into view multitudes innumerable of objects invisible before; and, for any thing experience has taught us, the number of stars may, to our powers of enumeration, be regarded as absolutely without limit.

§ 543. The mode of classification into orders is entirely arbitrary. Of a multitude of bright objects, differing in all probability intrinsically both in size and splendor and arranged at unequal distances, one must appear the brightest, another next below it, and so on. An order of succession must exist, and when it is gradual in degree and indefinite in extent, to draw a line of demarkation is matter of pure convention.

§ 544. Sir John Herschel proposes to make the scale of decreasing brightness of the stars which head the several orders of magnitudes, to vary inversely as the squares of the natural numbers, or as 1, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, $\frac{1}{25}$, &c.; that is, the brightest star of the first magnitude shall be *four* times that of the brightest of the second, *nine* times that of the brightest of the third, and so on: stars of intermediate brightness to be expressed decimally. Thus a star half way in brightness between the brightest of the third and of the fourth magnitudes would be expressed by $\frac{1}{(3.5)^2}$.

On the hypothesis that all the stars possess the same intrinsic brightness, coupled with the fact that the distance of the same luminous object varies inversely as the square root of its apparent brightness, the mere mention of the magnitudes of the stars would suggest, according to this classification, their relative distribution through space.

Fig. 95.



§ 545. To accomplish this photometrical classification, he proposes to receive the light from the planet Jupiter, at A, on the first face of a triangular prism, so as to fall on the second face at C under an angle of total reflection; this light, on its emergence from the third face, being received upon a convex lens D, would form an image of Jupiter's disk at F. An eye placed at E, within the field of the diverging waves, would receive the light from this image and that from a star proceeding along the line BE. The apparent brightness of Jupiter's image would vary inversely as the square of FE, because this planet has no sensible phases, and under the same atmospheric circumstances is of a constant brightness, while that of the star would be constant for all positions of the eye, and by altering the place of the latter the star and the image may be made to appear equally bright. The value of EF being ascertained for different stars, their relative brightness becomes known.

§ 546. Astronomers have generally agreed to restrict the first magnitude to about 23 or 24 stars, the second to 50 or 60, the third to about 200, and so on, their numbers increasing rapidly as we proceed in the order of decreasing brightness, the number of stars registered to include the seventh magnitude being from 12 to 15 thousand.

§ 547. Stars of the first three or four magnitudes are distributed pretty uniformly over the celestial sphere, the number being somewhat greater, however, especially in the southern hemisphere, along a zone following the course of a great circle through the stars called ε Orionis and α Crusis. But when the whole number visible to the naked eye are considered, they increase greatly towards the borders of the *milky way.* And if the telescopic stars be included, they will be found crowded beyond imagination along the entire extent of that remarkable belt and its branches. Indeed, *its whole light is composed of stars of every magnitude from such as are visible to the naked eye to the smallest point perceptible through the best telescopes.*

§ 548. The general course of the milky way, neglecting occasional deviations and following the greatest brightness, is that of a great circle inclined to the equinoctial under an angle of 63°, and cutting that circle in right ascension $0^h\ 47^m$ and $12^h\ 47^m$, so that its northern and southern poles are respectively in right ascension $18^h\ 47^m$ and $6^h\ 47^m$.

§ 549. This great circle of the celestial sphere with which the general course of the *milky way* most nearly coincides, is called the *gallactic circle.* To count the number of stars of all magnitudes visible in a single field of a telescope, and to alter the field so as to take in successively the entire celestial sphere, is to *gauge the heavens.*

§ 550. A comparison of many different gauges has given the average number of stars in a single field of 15′ diameter, within zones encircling the poles of the gallactic circle, found in the following

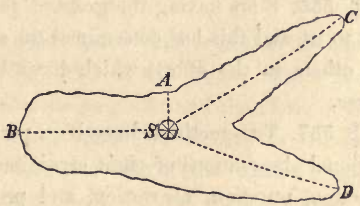
Table.

Zones of North Gallactic Polar distance.	Average Number of Stars in field of 15′.
0° to 15°	4.32
15 to 30	5.42
30 to 45	8.21
45 to 60	13.61
60 to 75	24.09
75 to 90	53.43

Zones of South Gallactic Polar distance.	Average Number of Stars in field of 15'.
0° to 15°	6.05
15 to 30	6.62
30 to 45	9.08
45 to 60	13.49
60 to 75	26.29
75 to 90	59.06

§ 551. This shows that the stars of our firmament, instead of being scattered in all directions indifferently through space, form a stratum of which the thickness is small in comparison with its length and breadth, and that our sun occupies a place somewhere about the middle of the thickness, and near the point where it subdivides into two principal laminæ, inclined under a small angle to one another. For to an eye so situated, the apparent density of stars, supposing them pretty equally scattered through the space they occupy, would be least in the direction $A\ S$, perpendicular to the laminæ, and greatest in that of its breadth $S\ B$, $S\ C$, or $S\ D$; increasing rapidly in passing from one direction to the other.

Fig. 96.



§ 552. For convenience of reference and of mapping, the stars are separated into groups by conceiving inclosing lines drawn upon the celestial sphere after the manner of geographical boundaries on the earth. The groups of stars within such boundaries are called *constellations*. The brightest star in each constellation is designated by the first letter of the Greek alphabet, the next brightest by the second, and so on till this alphabet is exhausted, when recourse is had to the Roman alphabet, and then to numerals. A star will be known from the name of the constellation and the letter or numeral: thus, α *Centauri*, 61 *Cygni*. Many of the brightest stars have also proper names, as *Sirius*, *Arcturus*, *Polaris*, &c.

§ 553. If, in Eq. (28), ρ denote the radius of the earth's orbit, π becomes the annual parallax, d the star's distance, and ω as before the number of seconds in radius unity. That equation gives

$$\frac{d}{\rho} = \frac{\omega}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (160)$$

§ 554. A line connecting the earth and a star would in the course of a year describe the entire surface of a cone of which the vertex would be the

star, and the base the orbit of the earth. The intersection of the *nappe* of this cone beyond the star with the celestial sphere would be an ellipse, and the apparent orbit of the star, arising from heliocentric parallax. The greater axis of this ellipse would be double the annual parallax.

§ 555. The stars floating, as it were, in space, and being subjected to the laws of universal gravitation, must each have a proper motion. In consequence of their vast distances from one another this motion may be comparatively slow, and their excessive distance from us almost conceals it, requiring years to describe spaces sufficiently great to subtend sensible angles at the earth. By comparing the relative places of stars at remote periods this proper motion has been detected and measured in a great many instances.

§ 556. Stars having the greatest proper motion are inferred to be nearest to us, and this has determined the selection of certain stars in preference to others in the efforts which have been made to ascertain their parallaxes.

§ 557. Two methods have been pursued. *First,* to find by careful meridional observations of right ascensions and declinations, cleared from refraction, nutation, aberration, and proper motion, the places of the star throughout the year, and thence the distance between those places most remote from one another. This is double the annual parallax.

Second, after selecting two stars very near to one another, and of which one has an obvious proper motion and the other not, to measure with the heliometer or micrometer their apparent distances apart, and to note the corresponding positions of the line joining them throughout the year; then to construct therefrom, after correcting for proper motion, the annual path of the moving star. Its longer axis will be double the annual parallax. This second is greatly the preferable method. The stars being separated by a few seconds only, they will be equally affected by refraction, nutation, and aberration, none of these depending upon actual distance. The method supposes the apparently immovable star to be immensely distant beyond the movable one.

By the first method Professor Henderson found the parallax of α Centauri to be $0''.913$; and by the second M. Bessel that of 61 Cygni to be $0''.348$.

§ 558. Assuming the parallax of α Centauri $= 1''$, to avoid multiplicity of figures, substituting it for π in Eq. (160), and writing the numerical value of ω, we have

$$\frac{d}{\rho} = \frac{\omega}{\pi} = 206265 \quad . \quad . \quad . \quad (161)$$

and in this proportion at least must the distance of the fixed stars exceed the distance of the sun from the earth.

Substituting for ρ its value, say in round numbers 95,000,000 of miles, and we have

$$d = 206265 \times 95000000 = 19595175000000^{m},$$

or about twenty billions of miles.

§ 559. Denoting the velocity of light by v, the time required for it to traverse the distance which separates the star from the earth by t, we have first, § 505,

$$v = 192000^{m},$$

and

$$t = \frac{d}{v} = 3^{y}.23\,;$$

that is to say, it would require light *three years and a quarter* to come from the nearest fixed star to the earth. And as this is the inferior limit which it is already ascertained that even the brightest and therefore, in the absence of all other indications, the nearest stars exceed, what is to be allowed for the distances of those innumerable stars of the smaller magnitudes which the most powerful telescopes disclose in the remote regions of the milky way ?

§ 560. The space penetrating power of a telescope, or the comparative distance to which a star would require to be removed in order that it may appear of the same brightness through the telescope as it did before to the naked eye, may be calculated from the aperture of the telescope as compared with that of the pupil of the eye, and from its power of reflecting or of transmitting incident light. The space penetrating power of the telescope employed on the gauge stars referred to in § 550 was 75. A star of the 6th magnitude removed to 75 times its distance would therefore still be visible, as a star, through that instrument, and admitting such a star to have 100th part the light of a standard star of the 1st magnitude, it will follow, from the law of illumination and distance, that such standard star if removed $75 \times 10 = 750$ times its distance would excite in the eye, when viewed through the telescope, the same impression as a star of the 6th magnitude does in the naked eye. Among the infinite number of stars in the remoter regions of the milky way it is but reasonable to conclude that there are many individuals intrinsically as bright as those which immediately surround us. The light of such stars must, therefore, have occupied $750 \times 3.25 = 2437.5$ years in travelling over the distance which separates them from our own system. And it follows that when we observe the places and note the appearances of such stars, we are only

reading their history more than two thousand years before. Nor is this conclusion, startling as it may appear, to be avoided without attributing an inferiority of intrinsic illumination to all the stars of the milky way—an alternative much less in harmony, as we shall see presently, with astronomical facts connected with other sidereal systems, revealed by the telescope, than are the views just taken.

§ 561. Of some of the stars whose parallaxes have been determined, the values of the parallaxes, and the names of the discoverers, are given in this

Table.

α	Centauri	$0''.913$;	Henderson.
61	Cygni	0.348;	Bessel.
α	Lyra	0.261;	Struve.
	Sirius	0.230;	Henderson.
1831	Groombridge . . .	0.226;	Peters.
ι	Ursæ Majoris . . .	0.133;	"
	Arcturus	0.127;	"
	Polaris	0.067;	"
	Capella	0.046;	"

§ 562. As remarked in the beginning of this chapter, the very best telescopes afford only negative information respecting the apparent diameters of the stars. The round and well-defined planetary disks which good telescopes exhibit are mere optical illusions, these disks diminishing more and more in proportion as the aperture and power of the instrument are increased. And the strongest evidence of a total absence of perceptible dimensions is the fact, that in occultations of the stars by the moon, the extinctions are ***absolutely instantaneous***.

If our sun were removed to the distance of α *Centauri*, its apparent diameter of $32' \; 3''$ would be reduced to only $0''.0093$, a quantity which no improvement of our present instruments can ever show with an appreciable disk.

§ 563. The star α *Centauri* has been directly compared with the moon by the method of § 545. By eleven such comparisons, after making due allowances for known sources of error, it was found that the light of the full moon exceeded that of the star in the proportion of 27408 to 1. Wollaston found the proportion of the sun's light to that of the moon to be as 801072 to 1. Combining these results, the light we receive from the sun is to that from α *Centauri* as 21,955,000,000, or about twenty-two thousand millions to one. Hence, the illumination being inversely as

the square of the distance, the intrinsic splendor of this star is to that of the sun as 2.3247 to 1. The light of *Sirius* is four times that of *α Centauri*, and its parallax only 0″.230, which give to Sirius a splendor equal to 140.2 times that of the sun.

§ 564. *Periodical Stars.*—Many of the stars, which in other respects are no way distinguished from the rest, undergo periodical increase and diminution of brightness, involving in one or two instances complete extinction and renovation. These are called *periodical stars.*

§ 565. The most remarkable star in this respect is *ο Ceti*, sometimes called ***Mira.*** It appears at variable intervals, of which the mean is 331^d 15^h 7^m. It retains its greatest brightness for a fortnight, being on some occasions equal to a large star of the second magnitude; decreases for about three months, becoming completely invisible to the naked eye for about five months, and increases for the remainder of the period. Such is the general course of its phases. It does not always return to the same degree of brightness, nor increase nor decrease by the same gradations, neither are the successive intervals of maxima equal. The mean interval is subject to a cyclical fluctuation embracing eighty-eight such intervals, and having the effect to shorten and lengthen the same about 25 days one way and the other.

§ 566. Another very remarkable periodical star is that called *β Persei*, and also frequently called *Algol.* It is usually visible as a star of the second magnitude, and as such continues for 2^d $13^h.5$, when it suddenly begins to diminish in splendor, and in about $3^h.5$ is reduced to the fourth magnitude, at which it continues for about 15^m. It then begins to increase, and in $3^h.5$ is restored to its usual brightness, going through all its changes in 2^d 20^h 48^m $58^s.5$. Recent observations indicate that this period is on the decrease, and not uniformly, but with an accelerated rapidity, indicating that it too has its cyclical period, and that instead of continuing to decrease, it will after a while be found to increase.

§ 567. The star *δ Cepheus* is also a periodical star. Its period from minimum to minimum is 5^d 8^h 47^m $39^s.5$. The extent of its variations is from the fifth to between the third and fourth magnitudes. Its increase is more rapid than its diminution—the former occupying 1^d 14^h, and the latter 3^d 19^h.

§ 568. The periodical star *β Lyra* has a period of 12^d 21^h 53^m 10^s, within which a double maxima and minima take place, the maxima being about equal, but the minima not. The maxima are about 3.4, and the minima 4.3 and 4.5. Here again the period is subject to change, which is itself periodical.

§ 569. Numerous other periodical stars are recorded. These remarkable variations of brightness, and the laws of their periodicity, have suggested the revolution of some opaque body or bodies around the stars thus distinguished, which, becoming interposed at inferior conjunction, would intercept a greater or less portion of the light on its way to the earth. Or the stars may possess very different degrees of intrinsic illumination on different portions of their surfaces, which, being subject to periodical changes and presented to the earth by an axial rotation of the stars, would produce the phenomena in question.

§ 570. *Temporary Stars.*—The irregularities above referred to may afford an explanation of other stellar phenomena, which have hitherto been regarded as altogether casual. Stars have appeared from time to time in different parts of the heavens blazing forth with extraordinary splendor, and after remaining a while, apparently immovable, have faded away and disappeared. These are called *temporary stars.* One of these stars is said to have appeared about the year 125 B. C., and with such brightness as to be visible in the daytime. Another appeared in A. D. 389, near α Aquilæ, remaining for three weeks as bright as Venus, and disappearing entirely. Also in 945, 1264, and 1572, brilliant stars appeared between Cepheus and Cassiopeia, which are supposed to be one and the same periodical star, with a period of 312, or perhaps 156 years. The appearance in 1572 was very sudden. The star was then as bright as Sirius; it continued to increase till it surpassed Jupiter, and was visible at mid-day. It began to diminish in December of the same year, and in March, 1574, it had entirely disappeared. So, also, on the 10th of October, 1604, a star not less brilliant burst forth in the constellation *Serpentarius*, which continued visible till October, 1605.

§ 571. Similar phenomena, though of less splendor, have taken place more recently. A star of the fifth magnitude, or 5.4, very conspicuous to the naked eye, suddenly appeared in the constellation Ophiuchus. From the time it was first seen it continued to diminish, without alteration of place, and before the advance of the season put an end to the observations upon it, had become almost extinct. Its color was ruddy, which was thought to have undergone many remarkable changes.

§ 572. The alternations of brightness of η Argûs are very remarkable. In 1677 it appeared as a star of the fourth, in 1751 of the second, in 1811 and 1815 of the fourth, in 1822 and 1826 of the second, in 1827 of the first, and in 1837 of the second magnitude. All at once, in 1838, it suddenly increased in lustre so as to surpass all the stars of the first magnitude except Sirius, Canopus, and α Centauri. Then it again diminished, but not

below the first magnitude, till April, 1843, when it had increased so as to surpass *Canopus*, and nearly equal *Sirius.*

§ 573. On careful re-examination of the heavens, and comparison of catalogues, many stars are missing.

§ 574. *Double Stars.*—Many of the stars when examined through the telescope appear double, that is, to consist of two individuals close together. They are divided into classes according to the proximity of their component individuals. The *first* class comprises those only of which the distance does not exceed 1″; the *second* those in which it exceeds 1″, but falls short of 2″; the *third* those in which it ranges from 2″ to 4″; the *fourth* from 4″ to 8″; the *fifth* from 8″ to 12″; the *sixth* from 12″ to 16″; the *seventh* from 16″ to 24″; and the *eighth* from 24″ to 32″.

Each of these classes is subdivided into two others, called respectively *conspicuous* and *residuary double stars.* The first comprehends those in which both individuals exceed the 8.25 magnitude, and are therefore separately bright enough to be seen with telescopes of very moderate capacity; the second embraces those which are below this limit of visibility. Specimens of each class will be found in the following

Table.

CLASS I.—0″ TO 1″.

γ Coronæ Bor.	η Coronæ.	7 Ophiuchi.	Atlas Pleiadum
γ Centauri.	η Herculis.	φ Draconis.	4 Aquarii.
γ Lupi.	λ Cassiopeiæ.	ϕ Ursæ Majoris.	42 Comæ.
ε Arietis.	λ Ophiuchi.	χ Aquilæ.	52 Arietis.
ζ Herculis.	π Lupi.	ω Leonis	66 Piscium.

CLASS II.—1″ TO 2″.

γ Circini.	ζ Bootis.	ξ Ursa Majoris.	2 Camelopardi.
δ Cygni.	ι Cassiopeiæ.	π Aquilæ.	32 Orionis.
ε Chamæleontis	ι_2 Cancri.	σ Coronæ Bor.	52 Orionis.

CLASS III.—2″ TO 4″.

α Piscium.	ν Virginis.	ζ Aquarii.	μ Draconis.
β Hydræ.	δ Serpentis.	ζ Orionis.	μ Canis.
γ Ceti.	ε Bootis.	ι Leonis.	ρ Herculis.
γ Leonis.	ε Draconis.	ι Trianguli.	σ Cassiopeiæ.
γ Coronæ Aus.	ε Hydræ.	κ Leporis.	44 Bootis.

CLASS IV.—4″ TO 8″.

α Crusis.	θ Phœnicis.	ξ Cephei.	μ Eridani.
α Herculis.	κ Cephei.	π Bootis.	70 Ophiuchi.
α Geminorum.	λ Orionis.	ρ Capricorni.	12 Eridani.
δ Geminorum.	μ Cygni.	ν Argus.	32 Eridani.
τ Coronæ Bor.	ξ Bootis.	ω Aurigæ.	95 Herculis.

CLASS V.—8″ TO 12″.

β Orionis.	ρ Antilæ.	*i* Orionis.
γ Arietis.	η Cassiopeiæ.	*f* Eridani.
γ Delphini.	θ Eridani.	2 Canum Ven.

CLASS VI.—12″ TO 16″

α Centauri.	ν Volantis.	κ Bootis.
β Cephei.	η Lupi.	8 Monocerotis.
β Scorpii.	ρ Ursæ Majoris.	61 Cygni.

CLASS VII.—16″ TO 24″.

α Canum Ven.	θ Serpentis.	24 Comæ.
ι Normæ.	κ Coronæ Aus.	41 Draconis.
ζ Piscium.	χ Tauri.	61 Ophiuchi.

CLASS VIII.—24″ TO 32″

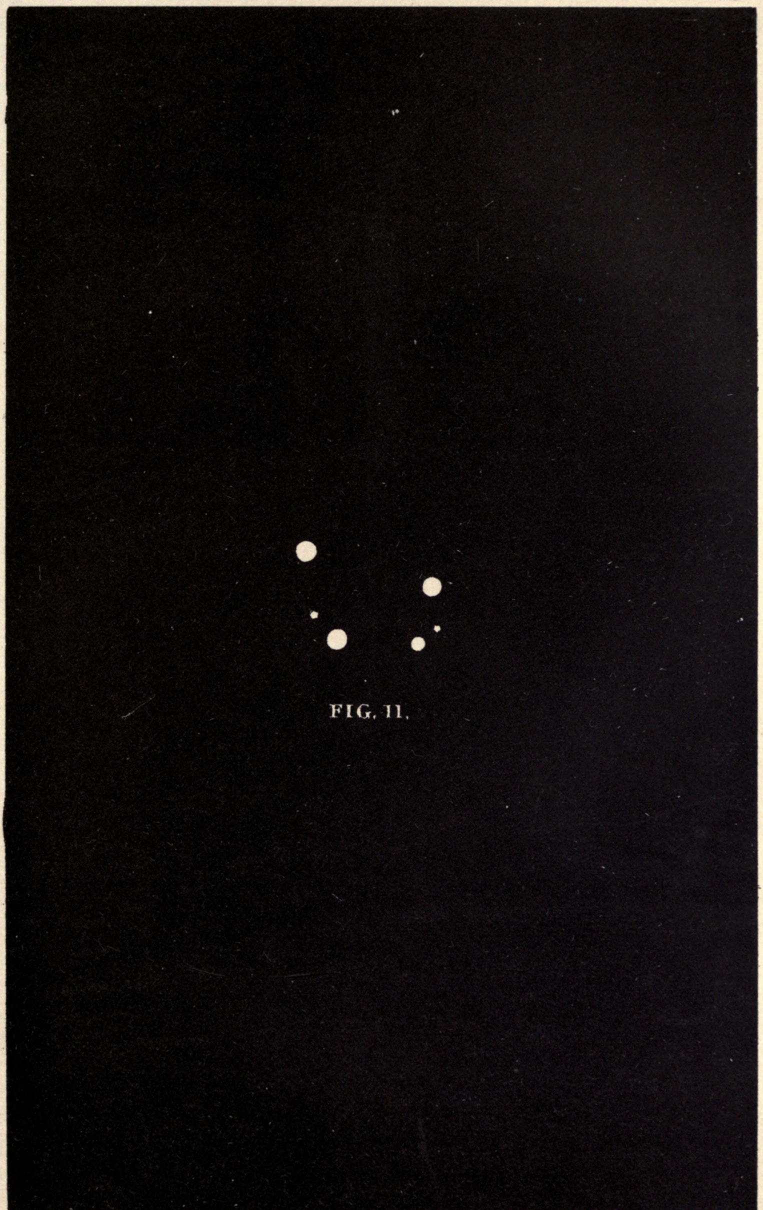
δ Herculis.	κ Herculis.	χ Cancri.
η Lyræ.	κ Cephei.	23 Orionis.
ι Cancri.	ψ Draconis.	

§ 575. *Triple, Quadruple, and Multiple Stars.*—Stars which answer to these designations also occur, and of them the most remarkable are,

α Andromedæ.	θ Orionis.	ξ Scorpii.
ε Lyra.	μ Lupi.	11 Monocerotis.
ζ Cancri.	μ Boötis.	12 Lyncis.

Of these, α *Andromedæ*, μ *Bootis*, and μ *Lupi*, appear through telescopes of considerable optical power only as ordinary double stars; and it is only when excellent instruments are used that their companions are subdivided and found to be extremely close double stars. ε *Lyra* offers the remarkable example of a double-double star. In telescopes of low power it appears as a coarse double star, but on increasing the power, each individual is perceived to be double, the one pair being about 2″.5, the other about 3″ apart. Each of the stars ζ *Cancri*, ξ *Scorpii*, 11 *Monocerotis*, and 12 *Lyncis*, consists of a principal star closely double and a smaller and more distant attendant; while θ *Orionis*, (Fig. 11, of plate,) presents four brilliant principal stars of the 4th, 6th, 7th, and 8th magnitudes, forming a trapezium, of which the longest diameter is 24″.4, and accompanied by two excessively minute and very close companions, to perceive both of which is one of the severest tests that can be applied to a telescope.

§ 576. Of the delicate subclass of double stars, or those consisting of very large and conspicuous double stars, accompanied by very minute companions, the following are specimens, viz.:



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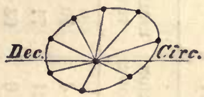
α_2 Cancri.	α Polaris.	κ Circini.	ϕ Virginis.
α_2 Capricorni.	β Aquarii.	κ Geminorum.	χ Eridani.
α Indi.	γ Hydræ.	μ Persei.	16 Aurigæ.
α Lyra.	ι Ursa Major.	7 Bootis.	91 Ceti.

§ 577. *Binary Stars.*—Many of the double stars are physically connected in such proximity to one another as to revolve about their common centre of gravity in regular orbits. These are called *binary stars.* They differ from what are called ordinarily "*double stars*" in being so near to one another as to be kept asunder only by a rotary motion about a common centre; whereas the individuals of a double star are separated by a vast distance, and appear double only in consequence of one being almost directly behind the other as seen from the earth.

§ 578. The position micrometer gives from time to time the apparent distance between the places into which the stars of a binary system are projected upon the celestial sphere, and also the angle which the arc of a great circle, drawn from one to the other, makes with the meridian passing through either, assumed as the central body; from these polar co-ordinates, the apparent orbit, as projected upon the celestial sphere, is easily traced.

Fig. 97.



§ 579. The relation which is found to connect the distances with the angular velocities shows the stars to be under the control of a central force, and the elliptical form of the orbit, with the eccentric position of the central star, is proof that this force can be no other than that of gravitation.

§ 580. Thus, the same principle which, under the influence of distance, directs the satellites about their primaries, and the primaries about our sun, also wheels distant *suns* around *suns*, each, perhaps, carrying with it its system of planets, and each planet a group of satellites.

§ 581. From the micrometrical measurements above referred to, and the intervals of time between them, the elements of the actual stellar orbits are easily computed.* A number of sets are given in the following table:

* See Memoirs of Royal Astronomical Society, vol. v. p. 171.

TABLE.

STARS.	APPARENT SEMI-AXIS.	ECCENTRICITY.	POSITION OF NODE.	PERIHELION FROM NODE.	INCLINATION.	PERIOD IN YEARS.	PERIHELION PASSAGE.	BY WHOM COMPUTED.
	″		° ′	° ′	° ′			
1. Herculis..........	1.189	0.44454	39 26	262 4	50 53	31.468	1829.50	Mädler.
2. η Coronæ B........	1.088	0.33760	24 18	261 21	71 8	43.246	1815.23	do.
3. ζ Cancri	1.292	0.23486	1 28	266 0	63 17	58.910	1853.37	do.
4. *a.* ξ Ursæ Maj......	3.857	0.41640	95 22	131 38	50 40	58.262	1817.25	Savary.
4. *b.* do........	3.278	0.37770	97 47	134 22	56 6	60.720	1816.73	Herschel, Jr.
4. *c.* do........	2.417	0.41350	98 52	130 48	54 56	61.464	1816.44	Mädler.
5. ω Leonis..........	0.857	0.64338	135 11	185 27	46 33	82.533	1849.76	do.
6. *a. p.* Ophiuchi	4.328	0.43007	147 12	125 22	46 25	73.862	1806.88	Encke.
6. *b.* do........	4.392	0.46670	137 2	145 46	48 5	80.340	1807.06	Herschel, Jr.
6. *c.* do........	4.192	0.44380	126 55	142 53	64 51	92.870	1812.73	Mädler.
7. Σ 3062	1.255	0.44958	15 3	137 27	35 31	94.765	1837.41	do.
8. ξ Bootis	12.566	0.59374	359 59	100 59	80 5	117.140	1779.88	Herschel, Jr.
9. δ Cygni............	1.811	0.60667	24 54	243 24	46 23	178.700	1862.87	Hind.
10. γ Virginis	3.580	0.87952	5 33	313 45	23 36	182.120	1836.43	Herschel, Jr.
11. *a.* Castor..........	8.086	0.75820	58 6	97 29	70 3	252.660	1855.83	do.
11. *b.* do............	7.008	0.79725	23 5	87 37	70 58	232.124	1913.90	Mädler.
11. *c.* do...........	6.300	0.24050	11 24	356 22	43 14	632.270	1699.26	Hind.
12. *a.* δ Coronæ B.....	3.918	0.69978	25 7	64 38	29 29	608.450	1826.60	Mädler.
12. *b.* do..........	5.194	0.72560	21 3	69 24	25 39	730.880	1826.48	Hind.
13. μ Bootis..........	3.218	0.84010	117 21	103 17	46 57	649.720	1852.50	do.
14. α Centauri	15.500	0.9500	86 7	291 22	47 56	77.000	1851.50	Jacob.

§ 582. If the annual parallax of the system, the apparent semi-axis of the stellar orbits, and the earth's radius vector, be substituted respectively for P, s, and ρ, in Eq. (29), d will become the number of linear units in the mean distance between the stars.

Assuming the data of the table, selecting α Centauri, and making $s = 15''.5$ and $P = 0.913$, we have

$$d = \frac{15''.5}{0.913} \cdot \rho = 16.977 \cdot \rho \quad . \quad . \quad . \quad . \quad . \quad (162)$$

whence the stellar orbits of α Centauri are (§ 422) about nine-tenths that of Uranus.

§ 583. Denoting by T the periodic time of a body about its centre, we have, *Analyt. Mechanics*, § 201,

$$T^2 = \frac{4\,\pi^2\,a^3}{k},$$

in which a is the mean distance, π the ratio of the circumference to diameter, and k the intensity of the central attraction at the unit's distance. For a second body

$$T'^2 = \frac{4\,\pi^2 a'^3}{k'};$$

whence

$$k : k' :: \frac{a^3}{T^2} : \frac{a'^3}{T'^2};$$

but from the laws of gravitation k and k' are directly proportional to the attracting masses, and we have

$$M : M' :: \frac{a^3}{T^2} : \frac{a'^3}{T'^2}.$$

Making $a = \rho$, $a' = d = 16.977\ \rho$; $T = 1$ year, and $T' = 77$ years; then will M denote the mass of the sun and M' that of the central star of α Centauri, and we have from the above proportion

$$M' = \frac{(16.977)^3}{77^2} \cdot M = 0.83 \cdot M;$$

that is, the mass of the central star is a little over eight-tenths that of our sun.

§ 584. *Color of Double Stars.*—Many of the double stars present the curious phenomena of complementary colors. In such instances the larger star is usually of a ruddy or orange hue, while the smaller one appears blue or green. The double star ι *Cancri* presents the beautiful contrast of

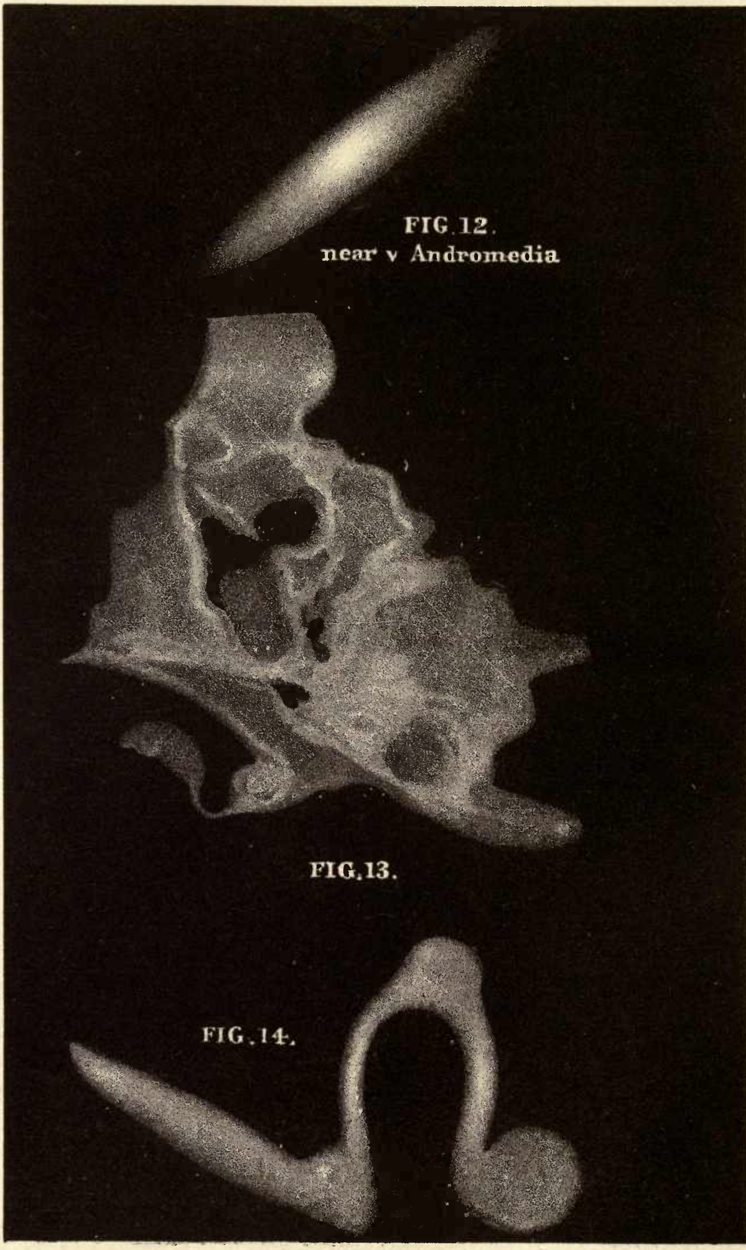
yellow and *blue; γ Andromedæ*, *crimson* and *green*. Where there is great difference in the magnitudes of the individuals, the larger is usually white, while the smaller may be colored; thus, η *Cassiopeiæ* exhibits the beautiful combination of a large white star and a small one of a rich ruddy purple. If this be not the mere optical effect of contrast of brightness, what variety of illumination *two suns*—a red and a blue one, a crimson and green one—must afford to the inhabitants of planets that circulate around them, having sometimes both suns above their horizon at once and at others each in succession, thus producing an alternation of red and blue, crimson and green days! Insulated stars of a red color, almost as deep as blood, occur in many parts of the heavens.

§ 585. *Proper Motions of the Stars.*—As might be expected from their mutual attractions, however enfeebled by distance and opposing attractions from opposite quarters, the stars are found to have a *proper motion*, which in the lapse of time has produced a sensible change of internal arrangement. Thus, from the time of Hipparchus, 130 years B. C., to A. D. 1717, eighteen hundred and forty-seven years, the conspicuous stars Sirius, Arcturus, and Aldebaran, are found to have changed their latitudes respectively 37′, 42′, and 33′, in a southerly direction. Besides, the observations of modern astronomy prove that such motions do really exist. The two stars 61 Cygni are found to have retained sensibly unchanged their distance apart for the last fifty years, while they have shifted their places in the heavens in the same interval no less than 4′ 23″, giving an annual proper motion to each of 5″.3. Of the stars not double, and no way differing from the rest in any other sensible particular, ε *Indi* and μ *Cassiopeiæ* have the greatest proper motions, amounting annually to 7″.74 and 3″.74 respectively.

§ 586. *Proper Motion of the Sun.*—The inevitable consequence of a proper motion in our sun, if not equally participated in by the rest, must be a slow *average* apparent tendency of all the stars to the point of the celestial sphere from which the sun is moving, and a corresponding retrocession from the opposite point—and this, however greatly individual stars may differ from such average by reason of their own peculiar proper motion. This is the necessary effect of parallax, and has been detected by observation.

By properly treating the observations on the stars of the northern hemisphere, the *solar apex*, as it is called, or the point towards which the sun was moving at the epoch of 1790, was in right ascension 250° 09′, and north polar distance 55° 23′. The southern stars gave, by a similar mode of treatment, right ascension 260° 01′, and north polar distance 55° 37′:

Plate IX.



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results so nearly identical as to remove all doubt of the sun's proper motion.

§ 587. All analogy would lead to the conclusion that the sun is describing an orbit of vast extent about the centre of gravity of the group of stars of which it forms a single member, and of which the milky way is to us but the distant trace, while this group may itself be moving as a single system around some other and vastly distant centre. A line drawn tangent to the solar orbit in 1790 pierced the celestial sphere near the stars π *Herculis* and α *Columba*, the sun being then moving towards the former and from the latter. And the result of calculations thus far gives to the sun a velocity of 422,000 miles a day, or little more than one-fourth the earth's rate of annual motion in its orbit.

NEBULÆ.

§ 588. Besides the stars which appear as shining points, there are cloud-like patches of light to be seen scattered here and there over the celestial vault. These are called *nebulæ.* They present themselves under great variety of shapes and sizes, as exemplified in Figs. 12, 13, 14 (fronting plate), and exhibit in the telescope different characters of internal structure with every increase of optical power. They are very unequally distributed over the heavens. In the northern hemisphere, the hours 3, 4, 5, 16, 17, and 18 of right ascension are singularly poor, while the hours 10, 11, and 12, especially the latter, are exceedingly rich in these objects. In the southern hemisphere a much greater uniformity prevails, with two remarkable exceptions, to be noticed presently. They have no decided tendency to any particular region.

§ 589. When viewed through the telescope, many nebulæ are resolved into stars, and the number that thus yield their cloud-like aspect increases with every augmentation of instrumental power. Nebulæ are therefore classified, with reference to their appearance through the telescope, into *resolvable*, *irresolvable*, *planetary*, and *stellar nebulæ*, and *nebular stars.*

§ 590. *Resolvable Nebulæ.*—These are usually called *clusters af stars.* Some are very broken in outline, while others are so regular as to suggest the prevalence of some internal action productive of symmetrical arrangement among their internal parts.

§ 591. *Irregular clusters* are much less rich in stars, and much less condensed towards the centre. In some the stars are nearly of the same size, in others very different. The group called the *Pleiades*, in which six

or seven stars may be counted with the naked eye, and fifty or sixty with the telescope, is one of the most obvious examples of this class. *Coma Berenices*, represented in Fig. 15, Plate X, is another such group.

§ 592. *Globular Clusters.*—These take their name from their round appearance. They are much more difficult of resolution, and some have frequently been mistaken for comets without tails. When viewed through the telescope, they are found to be composed of stars so crowded together as to occupy an almost definite outline, and to run up to a blaze of light towards the centre, where their condensation is greatest. It would be vain to attempt to count the stars in these clusters; some have been estimated to contain five thousand, within an area not greater than the tenth part of the lunar disk.

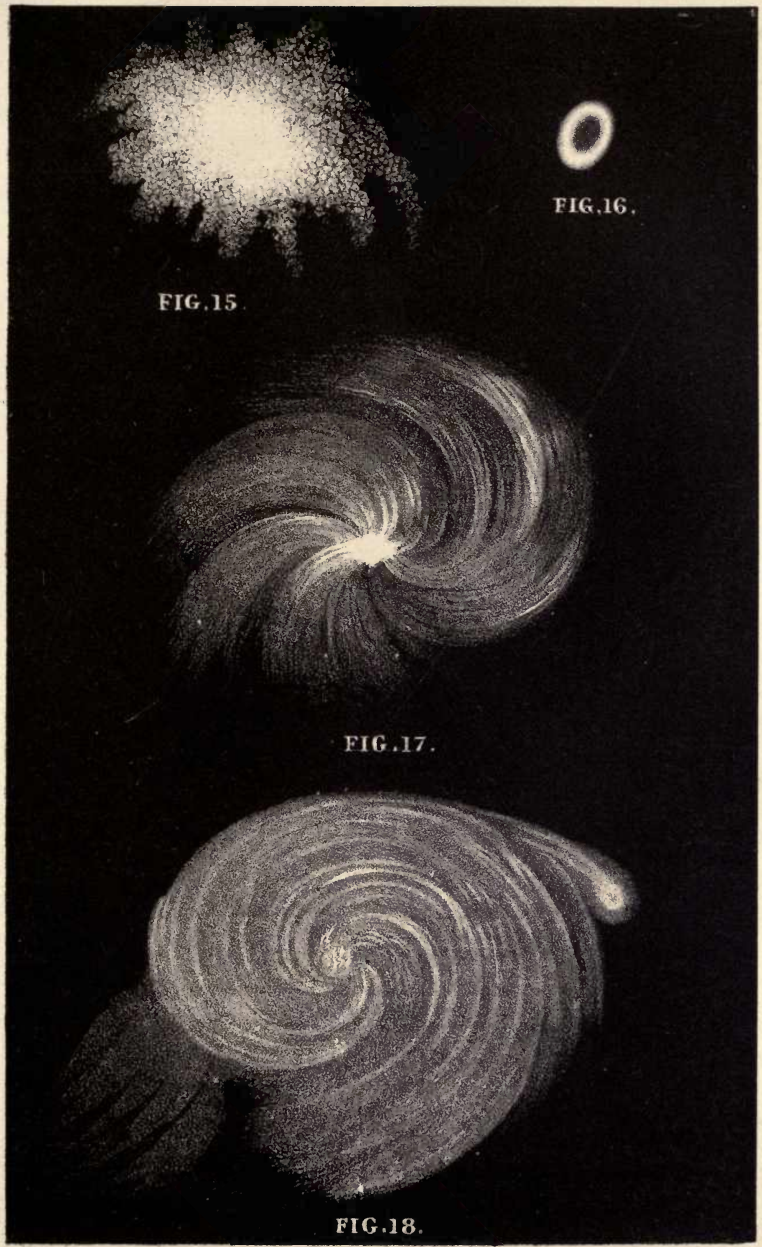
§ 593. *Elliptic Nebulæ.*—The figure here again suggests the name. They are of all degrees of eccentricity, from moderately oval to elongations so great as to be almost linear. In all, the density increases towards the centre, and generally their internal strata approach more nearly the spherical form than their external. Their resolvability is greater in the central parts; in some the condensation is slight and gradual, in others great and sudden.

The largest and finest specimen of elliptic nebulæ is in the Girdle of Andromeda, given in Fig. 12, Plate IX.

§ 594. *Annular nebulæ* also exist, but are very rare. The most conspicuous of this class is found between β and γ Lyræ, and may be seen through a telescope of moderate power. The central vacuity, Fig. 16, Plate IV, is not quite dark, but appears as a light-colored gauze stretched over a hoop. The powerful telescope of Lord Rosse resolves this nebula into excessively minute stars, and shows filaments of stars hanging to its edge.

§ 595. *Spiral Nebulæ.*—These are most curious objects. Their discovery is but very recent, and is due to the powerful instrument of Lord Rosse. As their name indicates, they appear to consist of a spiral or verticose arrangement of stars diverging from a centre, and suggest the idea of a vast self-luminous mass of matter, travelling to a common destination along separate curvilinear paths. Their form and general appearance are represented in Figs. 17 and 18, Plate X.

§ 596. *Planetary Nebulæ.*—These take their name from the planet-like disk which they present. In some instances they bear a perfect resemblance to a planet in this respect, being round or slightly oval, and quite sharply terminated. In some the illumination is perfectly equable: in others mottled, and of a peculiar texture, as if curdled. They are com-



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These nebulæ may be grouped into four great masses, which occupy the regions of *Orion*, of *Argo*, of *Sagittarius*, and of *Cygnus*.

§ 601. The *Magellanic Clouds*, or the *Nubeculæ* (Major and Minor), as they are called in celestial maps and charts, are two nebulous or cloudy masses of light conspicuously visible to the naked eye in the southern hemisphere, and in appearance and brightness resemble portions of the milky way of the same size. They are in shape somewhat oval, the larger deviating most from the circular form. The larger is situated between the hour circles $4^h\ 40^m$ and $6^h\ 40^m$, the parallels $156°$ and $162°$ north polar distance, and occupies an area of about 42 square degrees. The lesser, which is between the hour circles $0^h\ 28^m$ and $1^h\ 15^m$, and the parallels of $162°$ and $165°$ north polar distance, covers about ten square degrees. The general ground of both consists of large tracts of nebulosity in every stage of resolubility, from light irresolvable up to perfectly separated stars like the milky way, including groups sufficiently insulated and condensed to come under the designation of irregular and globular clusters, the latter being in every stage of condensation. In addition they contain nebular objects quite peculiar, and which have no analogy in any other part of the heavens. Globular clusters, except in one region of small extent, and nebulæ of regular elliptic forms are comparatively rare in the milky way, but are congregated in greatest abundance in parts of the heavens the most remote possible from the gallactic circle; whereas in the Magellanic Clouds they are indiscriminately mixed with the general starry ground.

§ 602. Regarding the *nubeculæ* as spherical in form, and not as vastly long vistas foreshortened by having their ends turned towards the earth—which would be improbable seeing there are two of them close together—the brightness of objects in their nearer portions cannot be much exaggerated, nor those in its remoter much enfeebled by difference of distance. It must, therefore, be an admitted fact that stars of the 7th and 8th magnitudes and irresolvable nebulæ may coexist within limits of distance comparatively small, and that all inferences in regard to relative distance drawn from relative magnitudes must be received with caution.

§ 603. *Our Sun a Nebulous Star.*—Various phenomena indicate that our sun is itself a nebulous star. The chief is that called the *zodiacal light*, which may be seen on any clear evening soon after sunset about the months of March, April, and May, and at the opposite seasons of the year just before sunrise, as a conically-shaped light, extending from the horizon upwards in the direction of the sun's equator. The apparent angular distance of its vertex V from the sun S varies from $40°$ to $90°$, and its breadth at its base, perpendicularly to its length, from $8°$ to $30°$. Every

Plate XII.

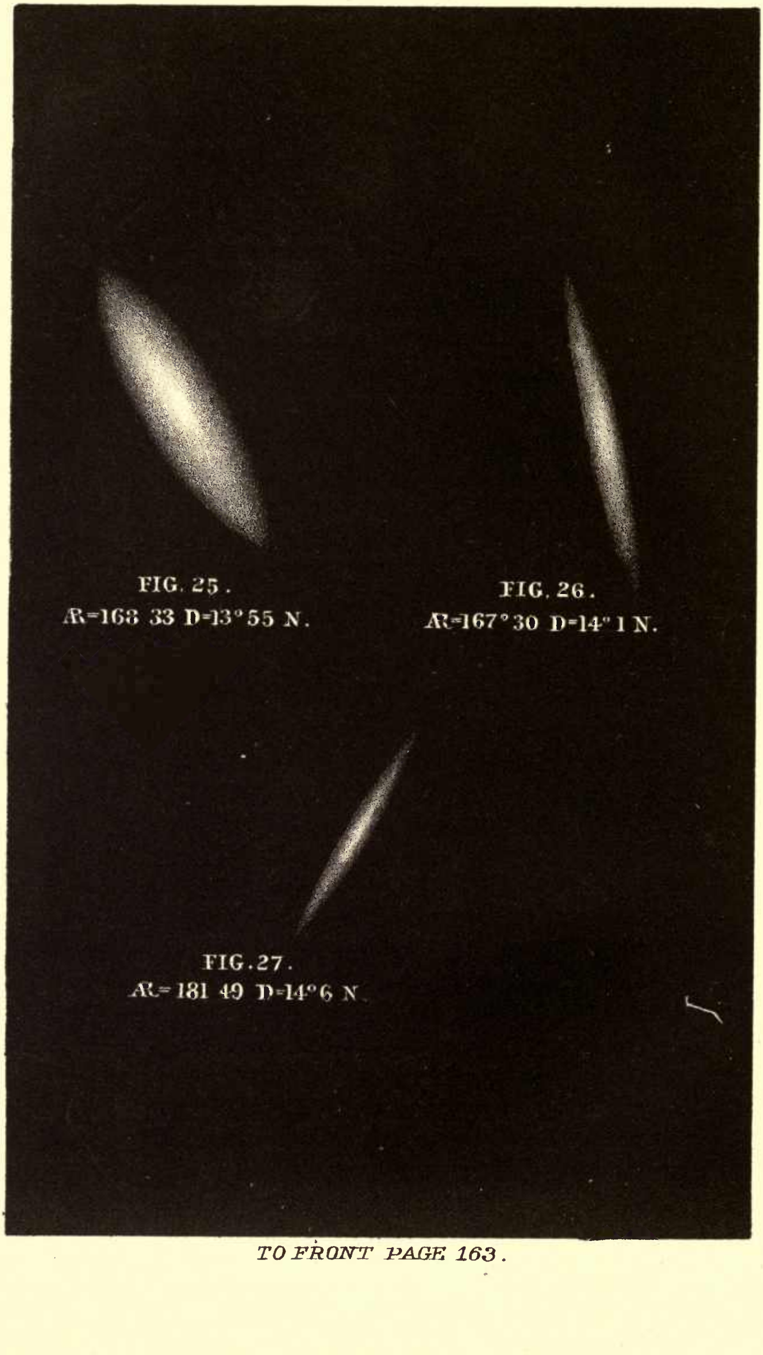

FIG. 25.
Æ=168 33 D=13°55 N.

FIG. 26.
Æ=167°30 D=14°1 N.

FIG. 27.
Æ=181 49 D=14°6 N.

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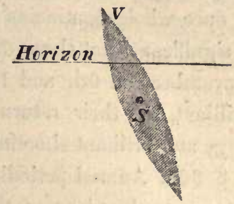
Fig. 98.



circumstance connected with it indicates it to be a lenticularly-formed envelope surrounding the sun, and extending beyond the orbits of Mercury and Venus and even to the Earth, its vertex having been seen 90° from the sun in a great circle. Different parts of the heavens furnish examples of similar forms. Figs. 25, 26, 27, Plate XII.

§ 604. *Aerolites.*—Nothing prevents that the particles of this vast material envelope may have tangible size and be at great distances apart, and yet compared with the planets, so called, be but as dust floating in the sunbeam. It is an established fact that masses of stone and lumps of iron, called *Aerolites*, do occasionally fall upon the earth from the upper regions of the atmosphere, and that they have done so since the earliest records. On the 26th April, 1803, one of these bodies fell in the immediate vicinity of the town of L'Aigle, in Normandy, and by its explosion into fragments, scattered thousands of stones over an area of thirty square miles. Four instances are recorded of persons having been killed by the descent of such bodies, and after every vain attempt to account for them as coming originally from the earth, and even from the moon, by volcanic projections, their planetary nature is now generally admitted. Their heat when fallen, the igneous phenomena which accompany them, their explosion on reaching the denser regions of our atmosphere, are accounted for by the condensation in front of them created by their enormous velocity, and by the relations of air, in a highly attenuated state, to heat.

§ 605. *Meteors.*—Besides these more solid bodies, others of much less density appear also to be circulating around the sun at the distance of the earth from that luminary. These on coming within the atmosphere appear as *shooting stars*, followed by trains of light, and are called *Meteors.* They appear now and then as great fiery balls, traversing the upper regions of the atmosphere, sometimes leaving long luminous trains behind them, sometimes bursting with a loud explosion, and sometimes becoming quietly extinct. Among these latter may be mentioned the remarkable meteor of August 18th, 1783, which traversed the whole of Europe, from Shetland to Rome, with a velocity of 30 miles a second, at a height of 50 miles above the earth, with a light greatly surpassing that of a full moon, and diameter quite half a mile. It changed its form visibly and quietly, separated into several distinct parts, which proceeded in parallel directions, each followed by a train.

§ 606. On several occasions meteors have appeared in astonishing

numbers, falling like a shower of rockets or flakes of snow, illuminating at once whole continents and oceans, even in both hemispheres. And it is significant that these displays have occurred between the 12th and 14th November and 9th and 11th August. In November they are much more brilliant, but their returns less certain than in August, when numerous large and brilliant shooting-stars with trains are almost sure to be seen.

§ 607. Annual periodicity, irrespective of geographical location, points at once to the place of the earth in its orbit as a necessary concomitant, and leads to the conclusion that at that place the earth enters a stratum, or annular stream of meteoric planets, in their progress of circulation around the sun. The earth plunging in its annual course into a ring of these bodies, and of such thickness as to be traversed in a day or two, their motions, referred to the earth as at rest, would be sensibly *uniform*, *rectilinear*, and *parallel*. Viewed from the centre of the earth, or from any point on its surface, neglecting the diurnal as being insignificant in comparison with the annual motion, their paths would appear to diverge from a common point on the celestial sphere. Now this is precisely what happens. The vast majority of the November meteors appear to describe arcs of great circles passing through γ Leonis, and those of August appear to move along paths having a common point in β Camelopardi.

§ 608 As the ring may have any position and be of an elliptical figure having any reasonable eccentricity, both the velocity and direction of each meteor may differ to any extent from those of the earth, so there is nothing in the great difference of latitude of these meteoric apices at all opposed to the foregoing conclusion.

§ 609. If the meteoric planets were uniformly distributed in the supposed ring, the earth's annual encounter with them would be certain if it occurred once; but if such ring be broken, and the bodies revolve in groups, with periods differing from that of the earth, years may pass without rencontre, and when such happen, they may differ to any extent in intensity of character, according as the groups encountered are richer or poorer in the number of their elements.

§ 610. From careful observations, made at the extremities of a base 50,000 feet long, it has been inferred that the heights of meteors at the instant of first appearance and disappearance, vary from 16 to 140 miles, and their relative velocities from 18 to 36 miles a second. Altitudes and velocities so great as these clearly indicate an independent planetary circulation round the sun.

§ 611. It is not impossible that some of these bodies may have been converted by the superior attraction of the earth, arising from greater prox-

imity, into permanent satellites; and there are those who believe in the existence of at least one of these bodies, which completes its circuit about the earth in about $3^h\ 20^m$, and therefore at a mean distance of about 5000 miles.

EPHEMERIDES.

§ 612. The facts and principles now explained enable us to predict the aspect of the heavens, or positions of the heavenly bodies, for all future time. This prediction is usually drawn up in the condensed form of tables, which are called *ephemerides.* The table relating to any one body is called the *ephemeris* of that body, as *the ephemeris of the sun*, *of the moon*, &c.

§ 613. Ephemerides are prepared in advance to subserve the wants and promote the interests of navigation, geography, and chronology, as well as of future astronomical discovery and research.

§ 614. To facilitate the computation of the ephemeris of a body, it is usual first to construct what are called its *tables;* and the manner of doing this may best be explained by taking a particular example, say that of the sun, or rather the earth, since this is the moving body; but as the place of the sun, as seen from the earth, differs from that of the earth as seen from the sun by the constant 180°, we shall speak of the sun.

§ 615. We have seen, § 197, how the mean longitude of the sun, his mean motion, longitude of the perigee, and eccentricity, may be found from observation and computation. These elements being found at epochs widely separated from one another, the changes which take place in the last three, and the rate of motion of the perigee, are ascertained.

§ 616. Having fixed upon any epoch, say mean noon or midnight, 1st January, 1800, any interval of time, either after or before the epoch, multiplied by the mean motion of the sun in longitude, will give the increase of mean longitude during that interval, and being added to the mean longitude at the epoch and the sum divided by 360°, the remainder will give the mean longitude at the beginning of the interval, if it be before, or end, if it be after the epoch. These longitudes, with the corresponding dates, being tabulated, give what is called a *table of epochs*, which tells by simple inspection the mean longitude on any given day, hour, minute, and second.

§ 617. The same process being performed with reference to the longitude of the perigee and its rate of change, gives a corresponding table in which the longitude of the perigee is found.

§ 618. Resuming Eq. (o), Appendix No. V., and causing $m\,t'$, which is the mean anomaly, to vary from 0° to 360°, corresponding equations of

the centre will result, and these properly arranged form a *table of equations of the centre*, of which the arguments, as they are called, are the mean anomalies. Then causing the eccentricity to vary according to ascertained rates, the same equation gives the elements of an additional table by which the equations of the centre may be corrected from time to time.

§ 619. Nutation causes the true equinox to oscillate about a mean place, its distance therefrom being equal to the algebraic sum of two functions, of which one depends upon the longitude of the moon's node, the other upon the longitude of the sun, and both upon the obliquity of the ecliptic. Tables containing the values of these functions for assumed places of the moon's node and of the sun, give the numbers whose sum is equal to the *equation of the equinoxes in longitude.*

§ 620. In addition, the larger of the planets, especially Venus and Jupiter, disturb the earth's orbit. These perturbations are computed by processes in physical astronomy, and their values arranged under heads that give the angular distances of the disturbing planets from the earth as seen from the sun, and, together with the place of the moon's node, furnish the argument with which other tables are entered that give the corresponding effects upon the sun's longitude.

§ 621. Lastly, as the purpose is to find the place where the sun's centre is to be seen, provision is made for the effect of aberration. This in the case of the sun is nearly constant, and equal to $-20''.25$, because of the small eccentricity of the earth's orbit, the greatest variation therefrom being less then $0''.35$. This constant is included in the epoch tables.

§ 622. *Ephemeris of the Sun.*—We are now prepared to find where the sun has been and where he will be on the celestial sphere throughout time. For this purpose, enter the table of epochs with the date, take out his mean longitude and the longitude of the perigee; the difference will be the mean anomaly, with which enter the table of the equations of the centre and take out the corresponding equation; add this to or subtract it from the mean longitude according to its sign, and the result will be the true longitude of the sun as affected by nutation and perturbations. Take these latter from the appropriate tables, and we have

True longitude of sun = mean longitude + equation of the centre + nutation or equation of equinoxes in longitude + perturbations.

§ 623. With the true longitude and obliquity of the ecliptic, we pass, by spherical trigonometry, § 149, to *right ascension* and *declination.*

§ 624. The mean anomaly in Eq. (n), Appendix V., gives the corresponding true anomaly; and the latter in Eq. (c), same Appendix, gives the

radius vector r, which in equations (28) and (29) give the corresponding horizontal parallax and apparent diameter.

§ 625. The mean longitude corrected for the equation of the equinoxes in right ascension, and diminished by the right ascension, gives the *equation of time.*

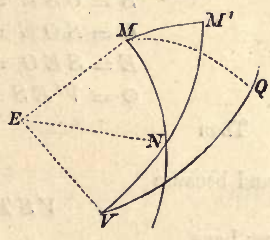
§ 626. These and other elements being determined at different epochs, say for every noon on some fixed meridian, their consecutive differences, divided by the number of hours between the epochs, give the hourly changes, and therefore the means of finding the value of the elements themselves for any other meridian.

The elements with their hourly changes make up, when properly tabulated, an *ephemeris of the sun.*

§ 627. *Ephemeris of the Moon.*—The motion of the moon is altogether more irregular and complicated than the apparent motion of the sun, owing mainly to the disturbing action of this latter body. But these and other perturbations have been computed and tabulated, and from these tables, including those of the node and inclination, the places of the moon in her orbit are found in much the same way as those of the sun in the ecliptic. The mean *orbit longitude* of the moon and of her perigee are first found and corrected: their difference gives her mean anomaly, opposite to which in the appropriate table is found the equation of the centre, and this being applied with its proper sign to the mean orbit longitude gives the true orbit longitude.

§ 628. Let E be the earth, M the moon, V the vernal equinox, VM' an arc of the ecliptic, VQ of the equinoctial, and MM' of a circle of latitude; then will MM' be the latitude and VM' the longitude of the moon, VN the longitude of the node and $VEN + NEM$ the orbit longitude of the moon.

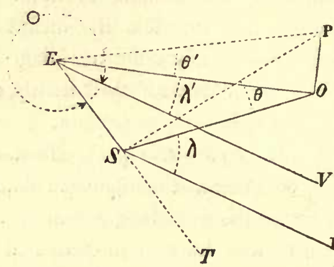
Fig. 99.



Subtracting from the orbit longitude of the moon the longitude of the node, the remainder NM will be the moon's angular distance from her node. This and the inclination MNM' will give, in the right-angled triangle MNM', the latitude MM' and the side NM', which latter added to the longitude of the node NV gives the longitude VM'. The latitude and longitude, together with the obliquity of the ecliptic, give, § 153, the right ascension and declination. The radius vector, equatorial horizontal parallax, apparent diameter, &c., are computed as in the case of the sun. And thus an ephemeris of the moon is constructed.

§ 629. *Ephemeris of a Planet.*—From tables of a planet its true orbit longitude *as seen from the sun* is found, as in the case of the moon as seen from the earth. From the heliocentric orbit longitude, heliocentric longitude of the node, and inclination, the heliocentric longitude and latitude, together with the radius vector, are found; just as the corresponding geocentric elements of the moon are found from similar data relating to the lunar orbit; and from the heliocentric longitude, latitude, and radius vector, we pass to the geocentric, thus:

Fig. 100.



§ 630. Let P be the planet, E the earth, S the sun, and O the projection of the planet upon the plane of the ecliptic. Draw from S and E the parallels SV and EV' to the vernal equinox, and make

$$r = ES \quad = \text{radius vector of earth;}$$
$$r' = SP \quad = \text{radius vector of planet;}$$
$$\lambda = VSO \quad = \text{heliocentric longitude of planet;}$$
$$\lambda' = V'EO = \text{geocentric longitude of planet;}$$
$$\theta = PSO \quad = \text{heliocentric latitude of planet;}$$
$$\theta' = PEO \quad = \text{geocentric latitude of planet;}$$
$$S = OSE \quad = \text{commutation;}$$
$$O = SOE \quad = \text{heliocentric parallax;}$$
$$E = SEO \quad = \text{elongation;}$$
$$\odot = V'ES = \text{longitude of sun.}$$

Then

$$SO = r' \cos \theta;$$

and because

$$VST = V'ES = 360° - \odot,$$

we have

$$S = 180° - (360° - \odot) - \lambda = \odot - 180° - \lambda;$$

whence the commutation is known. Then in the plane triangle OES,

$$r' \cos \theta + r : r' \cos \theta - r :: \tan \tfrac{1}{2}(E + O) : \tan \tfrac{1}{2}(E - O);$$

but

$$S + O + E = 180°,$$

whence

$$\tfrac{1}{2}(E + O) = 90° - \frac{S}{2} \quad . \; . \quad . \; . \; . \quad (163)$$

Substituting this above, we have

$$\tan \tfrac{1}{2}(E - O) = \cot \tfrac{1}{2} S \,.\, \frac{r' \cos\theta - r}{r' \cos\theta + r};$$

and making

$$\tan \chi = \frac{r' \cos\theta}{r},$$

$$\tan \tfrac{1}{2}(E - O) = \cot \tfrac{1}{2} S \,.\, \frac{\tan\chi - 1}{\tan\chi + 1} = \cot \tfrac{1}{2} S . \tan(\chi - 45^\circ) \;.\;. \quad (164)$$

Knowing from Eq. (163) the half sum of E and O, and from Eq. (164) their half difference, E and O become known.

And we have

$$\lambda' = E - (360^\circ - \odot) = E + \odot - 360^\circ \;.\;\;.\;\;. \quad (165)$$

§ 631. Again;

$$P\,O = E\,O \,.\, \tan\theta' = S\,O \,.\, \tan\theta;$$

whence

$$\frac{\tan\theta'}{\tan\theta} = \frac{S\,O}{E\,O} = \frac{\sin E}{\sin S};$$

and

$$\tan\theta' = \tan\theta \,.\, \frac{\sin E}{\sin S} \;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;. \quad (166)$$

From equations (165) and (166) the geocentric longitude and latitude become known.

§ 632. Denote by r'' the distance EP of the planet from the earth; then will

$$E\,O = r'' \cos\theta', \text{ and } S\,O = r' \cos\theta;$$

and in the triangle $E\,S\,O$

$$r'' \cos\theta' \;:\; r' \cos\theta \;::\; \sin S \;:\; \sin E;$$

whence

$$r'' = r' . \frac{\cos\theta}{\cos\theta'} \,.\, \frac{\sin S}{\sin E} \;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;. \quad (167)$$

The right ascension, declination, horizontal parallax, and apparent diameter, are found as in the case of the sun and moon.

§ 633. The ephemerides most commonly used in this country are those computed for the meridian of Greenwich, England, and published several years in advance under the title, "Nautical Almanac and Astronomical Ephemeris."

CATALOGUE OF STARS.

§ 634. Another important, indeed indispensable auxiliary to practical astronomy, is a *catalogue of stars.* This consists of a list of certain stars arranged in the order of their right ascensions, with the means of obtaining the right ascensions and declinations of the places in which they appear at any given epoch.

§ 635. By precession, § 157, nutation, § 156, and aberration, § 215, the right ascension and declination of a star are ever varying.

The place of a star referred to the mean equinoctial and mean equinox is called its *mean* place; that referred to the true equinoctial and true equinox, its *true* place; and that in which it is seen referred to the true equinoctial and true equinox, its *apparent* place.

The true place is equal to the mean, corrected for nutation; and the apparent place is equal to the true, corrected for aberration. The true and mean places are found from the apparent, by applying the same corrections, with their signs changed.

§ 636. The apparent places of the stars are used as points of reference on the celestial sphere; and knowing the right ascensions and declinations of these places, those of the apparent place of any other object become known also when the distance of the latter in right ascension and declination from one or more stars is found by instrumental measurement.

§ 637. *Annual Precession.*—The annual precession for any year is,

$$\begin{aligned}\text{Luni-solar} &= 50''.37572 - y \times 0''.0002435890,\\ \text{General} &= 50''.21129 + y \times 0''.0002442966;\end{aligned}$$

in which y denotes the number of the year from 1750, *minus* when *before* that epoch.

§ 638. The epoch of the catalogue which will be referred to hereafter, that of the British Association, is January 1st, 1850. Making $y = 100$, and denoting the nutation of obliquity by $\Delta\,\omega$, we have

$$\Delta\,\omega = 9''.2500 \cos \Omega - 0''.0903 \cos 2\,\Omega + 0''.0900 \cos 2\,☽ + 0''.5447 \cos 2\,\odot;$$

in which Ω denotes the mean longitude of the moon's node, ☽ the true longitude of the moon, and $\odot$ the longitude of the sun.

§ 639. And assuming the mean obliquity of the ecliptic for 1850 equal to $\omega = 23°\ 27'\ 31''$, we have then for the nutation in longitude, denoted by $\Delta\,L$,

$$\Delta\,L = -\,17''.3017 \sin \Omega + 0''.2081 \sin 2\,\Omega - 0''.2074 \sin 2\,☽ - 1''.2552 \sin 2\,\odot$$

§ 640. Denoting the equation of the equinoxes in right ascension by ΔA, we have

$$\Delta A = -15''.872 \sin \Omega + 0''.192 \sin 2\Omega - 0''.190 \sin 2 \text{☽} - 1''.500 \sin 2 \odot.$$

§ 641. Denoting the right ascension and declination of any body by α and δ respectively, and by p and p', its change in the same due to annual precession, then will

$$p = 46''.05910 + 20''.05472 \sin \alpha \,.\, \tan \delta \quad . \quad . \quad . \quad (168)$$

$$p' = 20''.05472 \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169)$$

§ 642. The change in right ascension and declination for any fractional portion of the year will be found by multiplying the above by

$$t = \frac{d}{365.25} = 0.00273785 \times d \quad . \quad . \quad . \quad . \quad (170)$$

In which d denotes the number of days from the beginning of the year to the end of the fraction.

§ 643. Denoting by $\Delta \alpha_{,}$ and $\Delta \delta_{,}$ the change in right ascension and declination arising from nutation, then, omitting terms involving $\sin 2 \text{☽}$, will

$$\left.\begin{aligned}\Delta \alpha_{,} = & -(15''.872 + 6''.888 \sin \alpha \,.\, \tan \delta) \,.\, \sin \Omega - 9''.250 \cos \alpha \,.\, \tan \delta \,.\, \cos \Omega \\ & + (0''.191 + 0''.083 \sin \alpha \,.\, \tan \delta) \,.\, \sin 2\Omega + 0''.090 \cos \alpha \,.\, \tan \delta \,.\, \cos 2\Omega \\ & - (1''.151 + 0''.500 \,.\, \sin \alpha \,.\, \tan \delta) \,.\, \sin 2\odot - 0''.545 \cos \alpha \,.\, \tan \delta \,.\, \cos 2\odot\end{aligned}\right\} (171)$$

$$\left.\begin{aligned}\Delta \delta_{,} = & \; 9''.250 \,.\, \sin \alpha \,.\, \cos \Omega - 6''.888 \cos \alpha \,.\, \sin \Omega \\ & - 0''.090 \sin \alpha \cos 2\Omega + 0''.083 \cos \alpha \,.\, \sin 2\Omega \\ & + 0''.545 \sin \alpha \,.\, \cos 2\odot - 0''.500 \cos \alpha \,.\, \sin 2\odot\end{aligned}\right\} \,.\, (172)$$

§ 644. *Aberration.*—Denoting by $\Delta \alpha_2$ and $\Delta \delta_2$ the change in right ascension and declination arising from aberration, disregarding the eccentricity of the earth's orbit,

$$\Delta \alpha_2 = -(20''.4200 \sin \odot \,.\, \sin \alpha + 18''.7322 \cos \odot \cos \alpha) \,.\, \sec \delta \quad . \quad . \quad (173)$$

$$\left.\begin{aligned}\Delta \delta_2 = & -(20''.4200 \sin \odot \,.\, \cos \alpha - 18''.7322 \cos \odot \sin \alpha) \sin \delta \\ & - 8''.1289 \cos \odot \cos \delta\end{aligned}\right\} \,.\, (174)$$

§ 645. Multiplying Eq. (168) by Eq. (170), adding together the product and equations (171) and (173), and denoting the apparent right ascension by α and the mean by α', there will result, after suitable reduction,

$$\begin{aligned}\alpha' - \alpha = \Delta \alpha = & \,(t - 0.343 \sin \Omega + 0.004 \sin 2\Omega - 0.025 \sin 2\odot) \times (46''.059 + 20''.055 \sin \alpha \tan \delta) \\ & - (9''.250 \cos \Omega - 0''.090 \cos 2\Omega + 0''.545 \cos 2\odot) \,.\, \cos \alpha \,.\, \tan \delta \\ & - 20''.420 \sin \odot \,.\, \sin \alpha \,.\, \sec \delta \\ & - 18''.732 \cos \odot \,.\, \cos \alpha \,.\, \sec \delta \\ & - 0''.0530 \sin \Omega + 0''.000 \sin 2\Omega - 0''.0039 \sin 2\odot.\end{aligned}$$

Multiplying Eq. (169) by Eq. (170), adding together the product and equations (172) and (174), and denoting the apparent declination by δ and the mean by δ', we also have, after reduction,

$$\begin{aligned}\delta' - \delta = \Delta\,\delta = & (t - 0.343 \sin \Omega + 0.004 \sin 2\,\Omega - 0.025 \sin 2\odot) \times 20''.055 \cos \alpha \\ & + (9''.250 \cos \Omega - 0''.090 \cos 2\,\Omega + 0''.545 \cos 2\odot) \sin \alpha \\ & - 20''.420 \sin \odot \,.\, \cos \alpha \,.\, \tan \delta \\ & - 18''.732 \cos \odot\, (\tan \omega \,.\, \cos \delta - \sin \alpha \,.\, \sin \delta).\end{aligned}$$

Neglecting the three last terms in the value for $\Delta\,\alpha$ as insignificant, and making

$$\begin{aligned}A &= -\,18''.732 \cos \odot,\\ B &= -\,20''.420 \sin \odot,\\ C &= t - 0.025 \sin 2\odot - 0.343 \sin \Omega + 0.004 \sin 2\,\Omega,\\ D &= -\,0''.545 \cos 2\odot - 9''.250 \cos \Omega + 0''.090 \cos 2\,\Omega,\end{aligned}$$

$$\begin{aligned}a &= \cos \alpha \,.\, \sec \delta,\\ b &= \sin \alpha \,.\, \sec \delta,\\ c &= 46''.059 + 20''.055 \sin \alpha \,.\, \tan \delta,\\ d &= \cos \alpha \tan \delta,\end{aligned}$$

$$\begin{aligned}a' &= \tan \omega \,.\, \cos \delta - \sin \alpha \,.\, \sin \delta,\\ b' &= \cos \alpha \,.\, \tan \delta,\\ c' &= 20''.055 \cos \alpha,\\ d' &= -\sin \alpha\,;\end{aligned}$$

the above become

$$\Delta\,\alpha = a\,.\,A + b\,.\,B + c\,.\,C + d\,.\,D \quad . \quad . \quad . \quad . \quad (175)$$

$$\Delta\,\delta = a'\,.\,A + b'\,.\,B + c'\,.\,C + d'\,.\,D \quad . \quad . \quad . \quad . \quad (176)$$

§ 646. *Proper Motion.*—To the foregoing must be added the proper motion of the star when it is known with sufficient accuracy, and is of sufficient magnitude to be taken into the account.

Equations (173) and (174) enable us to pass from the apparent to the true, or from the true to the apparent right ascension and declination of a star.

§ 647. Since the motion of the equinoxes is very slow, the values of the functions a, b, c, d, a', b', c', and d' will be sensibly constant for a number of years, particularly when the stars are not very near the poles, while those of the functions A, B, C, and D vary sensibly from day to day These latter are, therefore, computed for every day in the year, and their logarithms recorded in the astronomical ephemeris; the others are computed for the epoch of the catalogue, and their logarithms recorded opposite each star in the catalogue.

§ 648. *Construction of the Catalogue.*—The elements relating to each star occupy a portion of the two pages exposed to view on opening the catalogue. On the left-hand page will be found every thing relating to right ascension, and on the right, to declination. The left-hand page consists of eleven vertical columns: in the first is placed the number of the star, in the order of its right ascension; in the second, the name of the constellation in which it is situated, with its letter or number; in the third, its magnitude; in the fourth, its mean right ascension, January 1st, 1850, in time; in the fifth, its mean annual precession in right ascension, Eq. (168), reduced to time; in the sixth, its secular variation, reduced to time; in the seventh, its proper motion in right ascension, reduced to time; and in the eighth, ninth, tenth, and eleventh, the logarithms of the functions a, b, c, and d, reduced to time, respectively, each preceded by the sign of the function to which it belongs. The right-hand page consists of fifteen vertical columns, in the first of which the number of the star is repeated; the second contains the mean north polar distance, January 1st, 1850; the third, fourth, and fifth, the annual precession, secular variation, and proper motion in north polar distance, respectively; the sixth, seventh, eighth, and ninth, the logarithms of the functions a', b', c', and d' respectively, each preceded by the sign of the function to which it belongs; the remaining columns contain the numbers by which the star is recognized in the catalogues of the several authors, whose names are at the top.

Example.—Required the apparent right ascension and declination of γ Orionis, February 5th, 1854.

	h. m. s.		° ′ ″
Mean α January 1st, 1850 .	5 17 05.33	Mean N. P. D. . . .	83 47 25.7
4 years' prec. and pr. motion	+ 12.88	4 y'rs' prec. and pr. motion	− 14.9
Mean α January 1st, 1854 .	5 17 18.21	Mean N. P. D.	83 47 10.8

	Logs.	Nat. Nos.		Logs.	Nat. Nos.
a .	+ 8.0963		a' .	− 9.5120	
A .	− 1.1363		A .	− 1.1363	
aA .	− 9.2326 .	$-0^s.171$	$a'A$.	+ 0.6483 .	$+4''.449$
b .	+ 8.8188		b' .	− 8.3039	
B .	+ 1.1443		B .	+ 1.1443	
bB .	+ 9.9631	+ 0.919	$b'B$.	− 9.4482 .	− 0.281

	Logs.	Nat. Nos.		Logs.	Nat. Nos.
c .	$+ 0.5070$		c' .	$- 0.5721$	
C .	$- 9.2812$		C .	$- 9.2812$	
cC .	$- 9.7882$	$- 0.614$	$c'C$.	$+ 9.8533$.	$+ 0.713$
d .	$+ 7.1304$		d' .	$+ 9.9923$	
D .	$- 0.5713$		D .	$- 0.5713$	
	$- 7.7017$	$- 0.005$	$d'D$.	$- 0.5636$.	$- 3.661$
		$\Delta\alpha = + 0.129$			Δ N. P. D. $= + 1.220$

Hence app't right ascension, Feb. 5, 1854, $5^h\ 17^m\ 18^s.21 + 0^s.13 = 5^h\ 17^m\ 18^s.34$

app't N. P. D. . . . . $83°\ 47'\ 10''.80 + 1''.22 = 83°\ 47'\ 12''.02$

APPLICATIONS.

TIME OF CONJUNCTION AND OF OPPOSITION.

§ 649. To find from the ephemeris the time at which two bodies are in conjunction or opposition, find by inspection two simultaneous longitudes, one for each body, that differ by 0° or 180°. The corresponding time of the first will be that of conjunction, and of the second of opposition.

§ 650. But if these longitudes are not to be found in the tables, take therefrom two consecutive longitudes for each body, such, that those of the first shall differ from those of the second, in order, the least possible. Then, denoting the lesser and greater longitudes of the body having the greater velocity by l' and l'', those of the other by $l_{,}$ and $l_{,,}$ respectively, and the corresponding times by t' and t'', we have, because the longitudes of each are given for the same epochs,

$$(l'' - l') - (l_{,,} - l_{,}) : (t'' - t') :: l_{,} - l' : x,$$

whence

$$x = \frac{(l_{,} - l')\,(t'' - t')}{(l'' - l') - (l_{,,} - l_{,})};$$

in which x denotes the interval of time from t' to conjunction. And denoting the ephemeris time of conjunction by T_c, we have

$$T_c = t' + x = t' + \frac{(l_{,} - l')\,(t'' - t')}{(l'' - l') - (l_{,,} - l_{,})} \quad . \; . \; . \quad (177)$$

§ 651. Increasing the longitudes of one of the bodies by 180°, and se-

lecting those of the other to differ the least possible from these increased longitudes, then will T_e become the time of opposition.

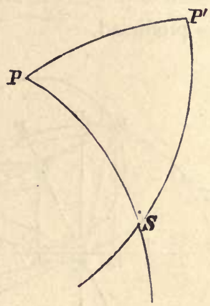
§ 652. T_e is the local time on the meridian for which the ephemeris is computed. Denoting the longitude, in time, of any other meridian west of this one by L, and the local time of conjunction or opposition by T, then will

$$T = T_e - L \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (178)$$

ANGLE OF POSITION.

§ 653. The angle made by a circle of latitude with a circle of declination through the centre of a body, is called the *angle of the body's position.*

Fig. 101.



To find this angle, let P be the pole of the ecliptic, P' that of the equinoctial, and S the centre of the body, and make

$\lambda = 90° - P\,S$ = latitude of the body;
$\delta = 90° - P'\,S$ = declination of the body;
$\varpi = P\,P'$ = obliquity of the ecliptic;
$S = P\,S\,P'$ = angle of position:

then will

$$\cos \varpi = \sin \lambda \,.\, \sin \delta + \cos \lambda \,.\, \cos \delta \,.\, \cos S,$$

and

$$\cos S = \frac{\cos \varpi - \sin \lambda \,.\, \sin \delta}{\cos \lambda \,.\, \cos \delta} \quad . \quad . \quad . \quad . \quad . \quad (179)$$

§ 654. If the body be the sun, then will $\lambda = 0$, and

$$\cos S = \frac{\cos \varpi}{\cos \delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (180)$$

PROJECTION OF A SOLAR ECLIPSE.

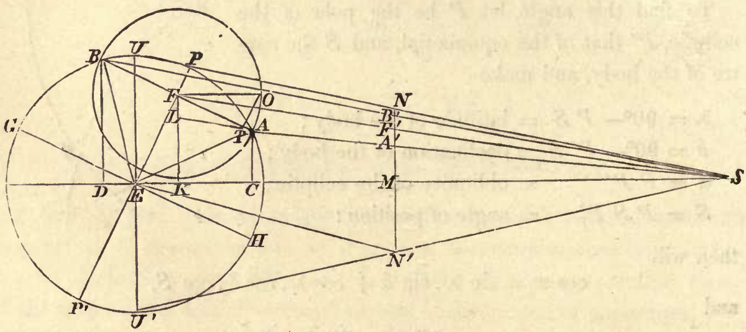
§ 655. A solar eclipse can take place only at new moon. Find the ephemeris time of the moon's conjunction with the sun. Then, by the method of interpolation, determine the sun's true longitude and hourly motion in longitude; the moon's true longitude and latitude, and hourly motion in longitude and latitude; the sun's and moon's horizontal parallaxes, and apparent semi-diameters, and the sun's angle of position.

§ 656. Conceive a cone tangent to the earth, and of which the vertex is at the sun. A section of this cone, by a plane between the earth and sun, will give an area upon which the sun's centre will appear tc be pro-

jected when viewed from different parts of the earth. A section at the distance of the moon from the earth, and perpendicular to the axis, is called the *circle of projection.*

§ 657. The diurnal rotation of the earth carries an observer once around his parallel of latitude in 24 hours; a line connecting him with the centre of the sun, describes an entire conical surface in the same time, and a section of this cone by the circle of projection will be the parallactic path of the sun as determined by the axial motion of the earth. This ellipse and the relative orbit of the moon, with a scale of time on each, indicating the simultaneous positions of the sun and moon, being constructed upon the plane of projection, all the circumstances of a local solar eclipse may easily be predicted.

Fig. 102.



§ 658. *Sun's Parallactic Path.*—Let $P\ G\ P'H$ be a meridian section of the earth by a declination circle through the sun's centre at S; E the earth's centre; P the elevated pole; $G\ H$ the projection of the equator; $B\ A$ that of the observer's parallel, and $N\ N'$ that of the circle of projection on the plane of the section. The projection $A'B'$, of $A\ B$ on the circle of projection by the lines $A\ S$ and $B\ S$, will be the conjugate, and that of the diameter of the parallel, which is perpendicular to $A\ B$, the transverse axis of the ellipse; the first being in and the second perpendicular to the declination circle through the sun.

§ 659. Make

$P = E\ N'U' =$ moon's horizontal parallax;

$\pi = E\ S\ U' =$ sun's " "

$l = A\ E\ H =$ reduced latitude of place;

$d = H\ E\ S =$ sun's declination;

$\rho = E\ A \quad =$ earth's radius;

$\omega =$ number of seconds in radius.

Draw $A\,C$, $B\,D$, and $F\,K$ perpendicular to $E\,S$, and we have

$$A\,C = \rho\,.\sin\,(l-d)\,; \qquad B\,D = \rho\,.\sin\,(l+d)\,;$$
$$E\,C = \rho\,.\cos\,(l-d)\,; \qquad E\,D = \rho\,.\cos\,(l+d).$$

Also, Eq. (28),

$$E\,S = \rho\,.\frac{\omega}{\pi}\,; \quad E M = \rho\,.\frac{\omega}{P}\,;$$

whence

$$S M = \rho\left(\frac{\omega}{\pi} - \frac{\omega}{P}\right) = \rho\,.\,\omega\,.\frac{P-\pi}{P\,\pi}.$$

From the figure,

$$S\,C = E\,S - E\,C = \rho\,.\frac{\omega}{\pi} - \rho\,.\cos\,(l-d).$$

Then in the triangles $S\,C\,A$ and $S\,M\,A'$,

$$S\,C : S\,M :: A\,C : A'M,$$

and by substitution

$$A'M = \rho\,.\frac{\sin\,(l-d)}{1 - \dfrac{\pi}{\omega}\,.\cos\,(l-d)}\,.\frac{P-\pi}{P}\,;$$

also,

$$S\,D = E\,S + E\,D = \rho\,.\frac{\omega}{\pi} + \rho\,.\cos\,(l+d)\,;$$

and in the same way as above, from the triangles $S\,D\,B$ and $S\,M\,B'$,

$$B'M = \rho\,.\frac{\sin\,(l+d)}{1 + \dfrac{\pi}{\omega}\,.\cos\,(l+d)}\,.\frac{P-\pi}{P}.$$

But π can never exceed $9''$, and ω is equal to $206264''.8$, so that the terms into which $\pi \div \omega$ enters as a factor may be neglected, and we have

$$M\,A' = \rho\,.\sin\,(l-d)\,.\frac{P-\pi}{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

$$M\,B' = \rho\,.\sin\,(l+d)\,.\frac{P-\pi}{P} \quad . \quad . \quad . \quad . \quad . \quad (182)$$

From which we see that the length of the projection of any dimension at the earth, and parallel to the circle of projection, is found by multiplying this dimension by $(P-\pi) \div P$.

§ 660. Denoting the conjugate axis $A'B'$ by $2\,b$, we have

$$2\,b = M\,B' - M\,A'\,;$$

and by substitution,

$$b = \rho \, . \cos l \, . \sin d \, . \frac{P - \pi}{P} \quad . \quad . \quad . \quad . \quad . \quad (183)$$

Also,

$$F A = \rho \, . \cos l \, ;$$

and because that diameter of the parallel of latitude, which is perpendicular to $A B$, is parallel to the plane of projection, we have, denoting the semi-transverse axis of the ellipse by a,

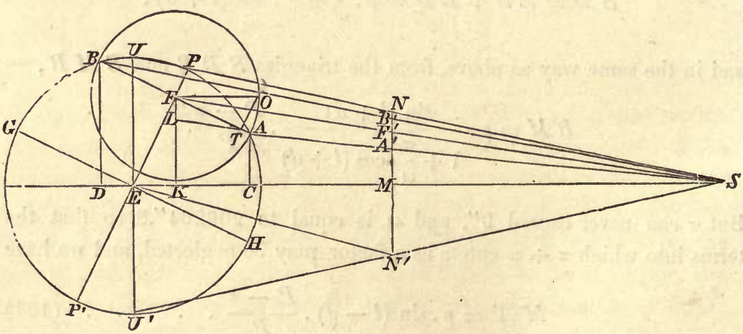
$$a = \rho \, . \cos l \, . \frac{P - \pi}{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad (184)$$

And denoting the distance $M F'$ from the centre of the circle of projection to that of the ellipse by Y, we have, taking half sum of equations (181) and (182)

$$Y = \rho \, . \sin l \, . \cos d \, . \frac{P - \pi}{P} \quad . \quad . \quad . \quad . \quad . \quad (185)$$

§ 661. Revolve the parallel of latitude about $A B$ till it coincides with the meridian section. When the observer is at A, it is to him apparent noon; when at B, apparent midnight; when at O, the angle $O F A$ is the apparent hour angle of the sun, and therefore local apparent time.

Fig. 102 bis.



Draw $O T$ perpendicular to $A B$, and $S L$ through the point T. The projection of $F L$ will give the distance of the sun from the transverse, and that of $O T$ his distance from the conjugate axis of his elliptical path. Denote the first by y, the second by x, and the hour angle $O F A$ by h. Then

$$F O = F A = \rho \, . \cos l \, ;$$

$$O T = \rho \, . \cos l \, . \sin h \, ;$$

$$F T = \rho \, . \cos l \, . \cos h \, ;$$

and since FLT is sensibly a right angle, the value of ESU, which is much greater than EST, never exceeding $9''$; and because $FTL = UEP = d$, we have

$$FL = FT \, . \sin d = \rho \, . \cos l \, . \cos h \, . \sin d;$$

and projecting FL and OT on the circle of projection, there will result

$$y = \rho \, . \cos l \, . \sin d \, . \cos h \, . \frac{P - \pi}{P} \quad . \quad . \quad . \quad . \quad (186)$$

$$x = \rho \, . \cos l \, . \sin h \, . \frac{P - \pi}{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (187)$$

But $\rho \div P$ is the linear subtense of the unit of arc in which P is expressed—say one minute. Calling this distance unity, equations (183), (184), (185), (186), and (187) may be written

$$b = \cos l \, . \sin d \, (P' - \pi') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (188)$$

$$a = \cos l \, . (P' - \pi') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (189)$$

$$Y = \sin l \, . \cos d \, . (P' - \pi') \quad . \quad . \quad . \quad . \quad . \quad (190)$$

$$y = \cos l \, . \sin d \, . \cos h \, . (P' - \pi') \quad . \quad . \quad . \quad (191)$$

$$x = \cos l \, . \sin h \, . (P' - \pi') \quad . \quad . \quad . \quad . \quad . \quad (192)$$

§ 662. Let C be the centre of the circle of projection, CN the trace of a circle of declination through the sun's centre on the plane of projection. Take the distance CF equal to Y, Eq. (190); through F draw AA' perpendicular to CN, and make $FA = FA' = a$, Eq. (189); take $FB = FB' = b$, Eq. (188); and making, successively, h equal to 15°, 30°, 45°, &c., in Eqs. (191) and (192), construct the corresponding hour

Fig. 108.

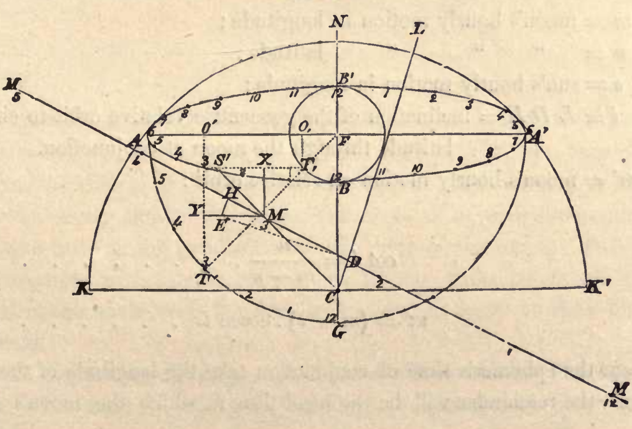
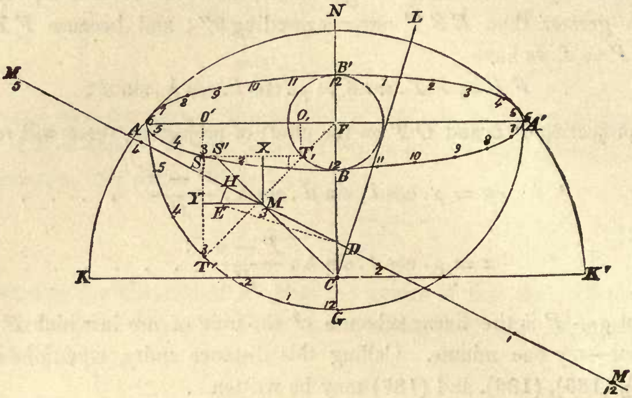


Fig. 103 bis.



points of the parallactic path of the sun's centre. The geometrical construction of this path is indicated in the figure.

§ 663. *Moon's geocentric relative orbit.*—Substitute in Eq. (180) the values of ϖ and δ, and make the angle NCL equal to the resulting value of S; the line CL will be the trace of a circle of latitude on the circle of projection. Make CD equal to the moon's latitude at conjunction, and draw DE perpendicular to CL and equal to the excess of the moon's hourly motion in longitude over that of the sun; draw EH perpendicular to ED and make it equal to the moon's hourly motion in latitude; through H and D draw an indefinite straight line; this line will represent the moon's geocentric relative orbit on the plane of the circle of projection.

§ 664. *Scale of time on the Moon's geocentric relative orbit.*—Make

m = moon's hourly motion in longitude;
n = " " " latitude;
s = sun's hourly motion in longitude;
$i = LDH$ = inclination of the geocentric relative orbit to circle of latitude through the moon at conjunction.
m' = moon's hourly motion on relative orbit:

then

$$\cot i = \frac{n}{m - s} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (193)$$

$$m' = (m - s) \, . \operatorname{cosec} i \quad . \; . \; . \; . \; . \; . \; . \quad (194)$$

From the ephemeris time of conjunction take the longitude of the place in time, the remainder will be the local time at which the moon's centre

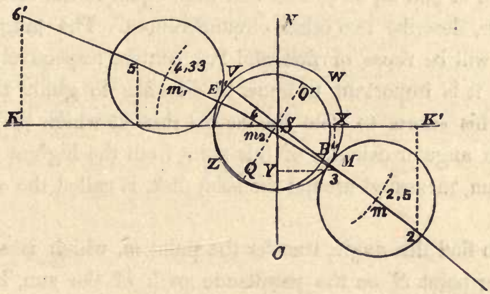
is at D. Let e denote its excess above the next preceding whole hour, and a the distance from D to the moon's centre at that hour; then will

$$a = (m - s) \,.\, \text{cosec}\, i \,.\, e \quad . \quad . \quad . \quad . \quad . \quad . \quad (195)$$

and laying off this distance from D to the west on the relative orbit, we have one point, and the entire hour on the scale of time indicating the positions of the moon at different times. From this point lay off distances to the west and east equal to the value of m' in Eq. (194), and there will result a series of points corresponding to the entire hours, as indicated in the figure. In the example before us, the local time of conjunction is about $2^h.33$ P. M., the hour point 2 falling about 0.33 of m' to the west of D.

§ 665. *Parallactic relative orbit of the Moon.*—The apparent path of the moon in reference to the sun's centre, as seen from the earth's surface, is the *moon's relative parallactic orbit.* To construct this path, draw through the sun's places on his parallactic path and the moon's places on her geocentric relative orbit, at the same hours, lines respectively parallel and perpendicular to CN, Fig. 103; a series of rectangles will thus be formed: the sides of these rectangles which terminate in the sun's places will be the co-ordinates of the moon's parallactic relative orbit, in reference to the sun's centre, regarded as fixed.

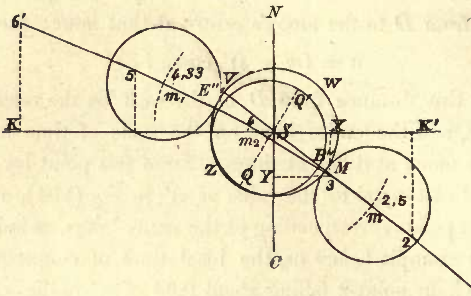
Fig. 104.



For example, draw $S\,Y$ and $M\,X$ parallel and $S\,X$ and $M\,Y$ perpendicular to CN; then making $S\,X$ and $S\,Y$, in Fig. 104, respectively equal and parallel to $S\,X$ and $S\,Y$ in Fig. 103, and drawing $X\,M$ and $Y\,M$ respectively parallel and perpendicular to $S\,N$, their intersection M will give a point in the parallactic relative orbit of the moon. This point in the example of the figure corresponds to 3^h. Other points being constructed in the same way, the orbit sought and its scale of time may be completed.

§ 666. *Beginning, Ending, and Greatest Obscuration.*—The hour intervals on the scale of time being suitably subdivided, with S as a centre

Fig. 104 bis.



and radius equal to the sum of the apparent semi-diameters of the sun and moon, describe an arc cutting the parallactic relative orbit of the moon in the points m and m_1; the corresponding numbers on the scale will be the times of beginning and ending, the former being at m and the latter at m_1. From S draw $S\,m_2$ perpendicular to the nearest portion of relative orbit, the number at m_2 on the scale will give the time of greatest obscuration. Also $Q\,Q'$ will be the quantity of the eclipse.

§ 667. *The Angle of First Contact.*—With S as centre and radius equal to the sun's apparent semi-diameter, describe the circumference $V\,W\,Z$; with m and m_1 as centres and radii equal to the moon's apparent semi-diameter, describe two other circumferences. The tangential points B'' and E'' will be those of first and last contact respectively. The former of these it is important to know in advance to guide the observer's attention in his efforts to note the actual time at which the solar eclipse begins. The angular distance of this point from the highest point or vertex of the sun, measured around the solar disk, is called the *angle of first contact.*

§ 668. To find this angle, transfer the point m, which is at $2^h.5$, to the corresponding point S' on the parallactic path of the sun, Fig. 103, and draw $C\,S'$. This line will represent the trace of a vertical circle through the sun's centre on the circle of projection, because every such circle must pass through the earth's centre, and therefore contain the point C. Making the angle $K'\,S\,V$ in Fig. 104, equal to $K'\,C\,S'$ in Fig. 103, the point V will be the vertex of the solar disk and $V\,S\,m$ the angle of first contact.

§ 669. By this construction, the beginning of a solar eclipse may be predicted within one minute of its actual occurrence, and this will in general be sufficient for the practical purpose of indicating the time to look for the instant of first contact—one of the most important elements, as we shall presently see, in the determination of terrestrial longitude.

For a full and complete investigation of the whole subject of eclipses, occultations, and transits, see Appendix XI., which consists entirely of the admirable paper of Mr. Woolhouse, first published as an appendix to the British Nautical Almanac for 1836.

PROJECTION OF A LUNAR ECLIPSE.

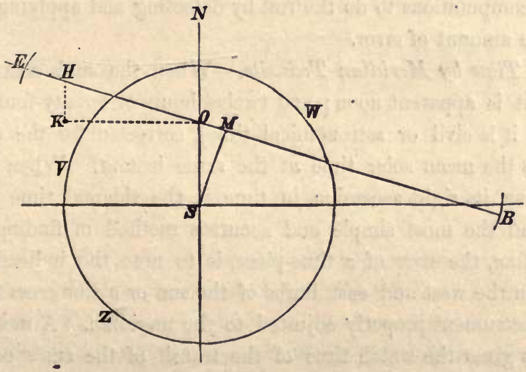
§ 670. During a lunar eclipse, the earth's shadow rests, as it were, upon the actual surface of the moon, and deprives it of a portion of the solar light which it would otherwise receive and reflect to a spectator on the earth.

§ 671. A section of the earth's shadow at the moon will have the same parallax as the moon, both being at the same distance from the earth; and regarding π as appertaining to the centre of this section, $P - \pi$ will be zero; Y, a, b, x, and y will, equations (188) to (192), be zero, and the parallactic path of the earth's shadow on the plane of projection will reduce to a point.

§ 672. Regarding, therefore, S, in Fig. 104, as the centre of a section of the earth's shadow at the moon, and $V\,W\,Z$ as its circumference, then will $m\,m_2\,m_1$ represent, not the parallactic, but the geocentric relative orbit of the moon.

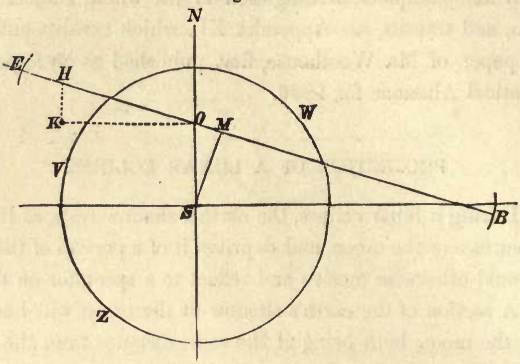
§ 673. Hence, to project a lunar eclipse, find from the ephemeris the time of opposition or full moon, the corresponding values of the moon's latitude, hourly motion in latitude, longitude, hourly motion in longitude, horizontal parallax, and apparent semi-diameter; also the sun's longitude, hourly motion in longitude, horizontal parallax, and apparent semi-diameter.

Fig. 105.



§ 674. Then, with S as centre and radius equal to that of the earth's shadow at the moon, Eq. (148), describe the circumference $V\,W\,Z$. Draw

Fig. 105 bis.



$S N$ to represent an arc of a circle of latitude; make $S O$ equal to the moon's latitude at opposition, and construct the moon's relative geocentric path and scale of time, as in § 663–4. With S as centre and radius equal to that of the earth's shadow, increased by the moon's apparent semi-diameter, describe an arc cutting the relative orbit in B and E, and let fall from S the perpendicular $S M$ on the relative orbit. The numbers on the scale of time at B, M, and E will give respectively the time of beginning, middle, and ending of a lunar eclipse.

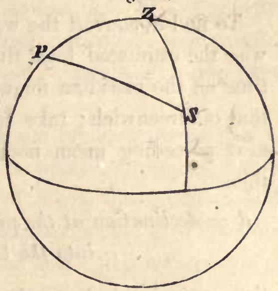
TIME OF DAY.

§ 675. The imperfection incident to all machinery makes it impossible to construct a time-piece to run accurately to mean solar or sidereal time. The best efforts result only in approximations; and when these are made to the utmost attainable limits, it must remain for astronomical observations and computations to do the rest by detecting and applying from time to time the amount of error.

§ 676. *Time by Meridian Transits.*—When the sun's centre is on the meridian it is apparent noon; and twelve hours or twenty-four hours (according as it is civil or astronomical time), corrected for the equation of time, gives the mean solar time at the same instant. When a star is on the meridian, its right ascension, in time, is the sidereal time at that instant. And the most simple and accurate method of finding the time, and, therefore, the error of a time-piece, is to note the indications of the latter when the west and east limbs of the sun or a star cross the wires of a transit instrument properly adjusted to the meridian. A mean of these indications gives the watch time of the transit of the sun's centre in the first case, or that of the star in the second. The difference between the first and the mean solar time gives the error on mean solar time; and

between the second and the star's right ascension, the error on sidereal time.

Fig. 106.



It is not, however, always possible to use the transit, and recourse must be had to observations off the meridian.

§ 677. *Solar Time by a single Altitude.*—To find the mean solar time and error of time-keeper from one observed altitude of the sun: Let P be the pole, Z the zenith, and S the place of the sun's centre; and make

$a =$ true altitude of sun's centre $= 90° - ZS$;
$d =$ true declination of sun . $= 90° - PS$;
$l =$ latitude of the place . . $= 90° - PZ$;
$P =$ hour angle ZPS;
$\varepsilon =$ error of the watch;
$T_w =$ watch time of observation;
$T_a =$ apparent time of observation;
$T_m =$ mean time of observation;
$E =$ equation of time.

Then, in the triangle PZS, we have from spherical trigonometry,

$$\text{Sin } \tfrac{1}{2} P = \pm \sqrt{\frac{\cos(S+l) \cdot \cos(S+d)}{\cos l \cdot \cos d}};$$

in which

$$S = \frac{270° - (a + d + l)}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (196)$$

whence

$$P = 2\sin^{-1}\left(\pm \sqrt{\frac{\cos(S+l) \cdot \cos(S+d)}{\cos l \cdot \cos d}}\right) \quad . \quad . \quad (197)$$

$$T_a = \tfrac{1}{15} P;$$

$$T_m = T_a \pm E = \tfrac{1}{15} P + E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (198)$$

and

$$\varepsilon = T_m - T_w = \tfrac{1}{15} P + E - T_w \quad . \quad . \quad . \quad . \quad . \quad (199)$$

The latitude of the place of observation is supposed known. The value of ε requires, in addition, the values of a, d, E, and T_w to be known.

To find a and T_w, measure with a sextant and artificial horizon, or other instrument for taking altitudes, the altitude of the upper or lower limb of the sun—say the lower—and note the precise indication of the watch at the instant. T_w is thus found, and

a = *observed altitude of the lower limb — refraction + apparent semi-diameter of the sun + the sun's parallax in altitude.*

To find d, convert the watch time—supposed not greatly in error, otherwise the estimated local time of observation—into the corresponding local time on the meridian for which the ephemeris of the sun is computed, say that of Greenwich; take from the ephemeris the sun's declination for the next preceding mean noon, and also the hourly change in declination; then

d = *declination at the preceding noon* $\pm$ *its hourly change multiplied into the Greenwich time of observation;*

the upper sign to be used when the declination is increasing, and the lower when decreasing.

To find E, take from the ephemeris the equation of time for the next preceding mean noon, and the hourly change; then

E = *equation at the preceding mean noon* $\pm$ *its hourly change into the Greenwich time of observation.*

§ 678. It is usual to avoid the correction for semi-diameter by clamping the instrument at some assumed altitude, and noting the time, by the watch, that the upper and lower limb of the sun attain this altitude. The mean of these times will be the time when the sun's centre had this same altitude, and it will only be necessary to correct the observed altitude for refraction and parallax.

§ 679. It is to be remembered that P has, Eq. (197), the double sign: the positive answers to the case in which the hour angle is west, or the observation is made in the afternoon; and the negative to that in which the hour angle is east, or the observation is made in the morning. In this latter case, $\frac{1}{15} P$ must be replaced by $12^h - \frac{1}{15} P$ if civil, or $24^h - \frac{1}{15} P$ if astronomical time be sought.

This process for finding the error of a time-piece is called the method of *single altitudes.*

§ 680. *Sidereal time by a single Altitude of a Star.*—Proceed exactly as in § 677–9, using the declination and observed altitude of the star for those of the sun, correcting the altitude for refraction only, and find the value of $\frac{1}{15} P$, to which add the right ascension of the star as found from the catalogue; the sum will be the sidereal *time.*

§ 681. The rules for converting solar into sidereal time, and the reverse, together with tables for facilitating the same, are given in the solar ephemeris.

§ 682. *Time of Sunrise and Sunset.*—At the instant of apparent sunrise, the sun's centre is in the horizon; the apparent altitude of its lower limb is equal to minus its apparent semi-diameter, and a, in Eq. (197), becomes the difference between the horizontal refraction and parallax. Making this substitution in Eq. (197) we find P, and this in Eq. (198) gives T_m.

§ 683. *Time by Equal Altitudes.*—If the sun retained unchanged his declination, equal altitudes would correspond to equal hour angles, and the half sum of the watch times, augmented by 6^h when the dial-plate is divided into 12, and 12^h when divided into 24 hours, would give the watch time of apparent noon. Twelve or twenty-four hours, depending upon the dial-plate, corrected for the equation of time would give the mean time of apparent noon, and the difference between this and the corresponding watch time would give the error.

But the sun is ever changing his declination, and when the effect of the change is to lessen his distance from the elevated pole between the observations, his hour angle in the morning will be less than in the afternoon at equal altitudes; the watch time of apparent noon, as above found, would be too late, and must be corrected by subtracting therefrom half the excess of the evening over the morning hour angle. Conversely, when the effect is to augment the distance from the elevated pole, the morning hour angle will exceed the evening; the watch time of apparent noon, as found by the rule, will be too early, and must be augmented by half the excess of the morning over the evening hour angle.

Denoting this correction by $t_{\prime}$, its value in seconds of time will, Appendix XII., be given by

$$t_{\prime} = -\delta \,.\, \tan d \,.\, \frac{t}{1440 \tan 7\frac{1}{2}t} + \delta \,.\, \tan l \,.\, \frac{t}{1440 \sin 7\frac{1}{2}t};$$

or making

$$\frac{t}{1440 \sin 7\frac{1}{2}t} = A,$$

$$\frac{t}{1440 \tan 7\frac{1}{2}t} = B;$$

$$t_{\prime} = +A \,.\, \delta \,.\, \tan l - B \,.\, \delta \,.\, \tan d \quad . \quad . \quad . \quad . \qquad (200)$$

In which

$t =$ the interval of time between the observations in hours;
$l =$ latitude of place;
$d =$ declination of sun at noon of the day,

δ = double daily variation in declination, or change from noon of preceding to noon of following day.

The value of t will be subtractive for apparent noon, and additive for apparent midnight, when δ is positive, and conversely.

The logarithms of the values of A and B are given in Table IV., for every two minutes, from two hours up to twenty-three; the latitude of the place must be known; the declination is found as in § 677, and δ is obtained from the ephemeris.

§ 684. *Change of atmospheric temperature and of pressure.*—In what precedes it is supposed that when the measured altitudes are equal, the true altitudes are so likewise; but this depends upon the state of the air remaining the same between the observations. If the barometer and thermometer vary, the refraction will vary, and the true altitudes will be unequal when the observed are equal. A further correction becomes therefore necessary, and its value is the increment or decrement of the hour angle of the sun, which would change his altitude by a quantity equal to the difference between the morning and evening refraction. The amount of this correction, denoted by $t_{\prime\prime}$ and expressed in seconds of time, is, Appendix XIII., given by

$$t_{\prime\prime} = \tfrac{1}{15} \cdot \frac{(r' - r) \cdot \cos a}{\cos l \cdot \cos d \cdot \sin P} \quad . \quad . \quad . \quad . \quad . \quad (201)$$

in which

r = morning refraction, in seconds of arc;
r' = evening refraction, in seconds of arc;
P = half the interval between the observations in arc;
a = altitude of sun;
d = declination;
l = latitude of place.

This process for finding the time of day, or error of a time-piece, is called the method of *equal altitudes;* and the value of $t_{\prime\prime}$ in Eq. (200), is called the *equation of equal altitudes.*

§ 685. The altitudes should be taken on or near the prime vertical, since in that position the altitudes change most rapidly.

AZIMUTHS.

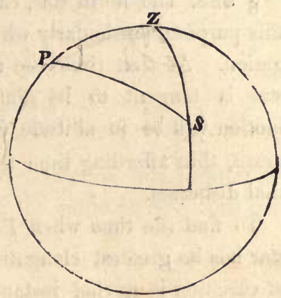
§ 686. In surveys and geodetic operations, it is necessary to determine the bearings of objects in reference to the meridian of the station from which they are seen. These bearings are measured by the angles at the

zenith included between the vertical circles through the objects and the meridian.

§ 687. *True Bearing.*—To find the true bearing of an object from a given station. Take the instrumental bearing of the object and of some heavenly body, and add their difference to the true azimuth of the body; the sum will be the true azimuth of the object.

To find the true azimuth of a heavenly body, note the time its instrumental bearing is taken.

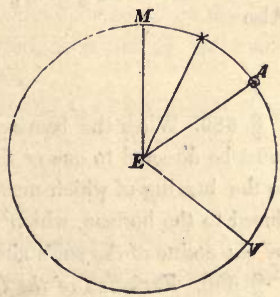
Fig. 107.



Let Z be the zenith, P the pole, and S the heavenly body, say the sun or a star; make

$P =$ hour angle $Z\,P\,S$;
$l =$ latitude of place;
$d =$ declination of body;
$T_m =$ mean solar time of observation;
$E =$ corresponding equation of time;
$T_a =$ apparent solar time of observation;
$T_* =$ sidereal time of observation;
$A_\odot =$ right ascension of mean sun;
$A_* =$ right ascension of a star.

Fig. 108.



Then with the sun and mean solar time, we have

$$P = 15 \,.\, T_a = 15\,(T_m \pm E) \quad (202)$$

With the mean solar time and star,

$$P = 15\,(T_m + A_\odot - A_*) \qquad (203)$$

With the sidereal time and the sun,

$$P = 15\,(T_* - A_\odot \pm E) \;.\;.\; (204)$$

With the sidereal time and star,

$$P = 15\,(T_* - A_*) \;.\;.\;.\;.\; (205)$$

Also make

$\varphi = Z\,P = 90^\circ - l$;
$\delta = P\,S = 90^\circ - d$;
$A =$ the angle $P\,Z\,S = 180^\circ -$ azimuth of the body;
$\xi =$ " " $Z\,S\,P$.

Then in the triangle $Z\,P\,S$, from *Napier's Analogies*,

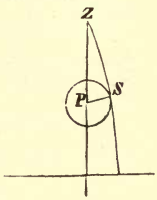
$$\tan \tfrac{1}{2}\,(A + \xi) = \cot \tfrac{1}{2} P \,.\, \frac{\cos \frac{1}{2}\,(\delta - \varphi)}{\cos \frac{1}{2}\,(\delta + \varphi)} \quad . \quad . \quad . \quad (206)$$

$$\tan \tfrac{1}{2}\,(A - \xi) = \cot \tfrac{1}{2} P \,.\, \frac{\sin \frac{1}{2}\,(\delta - \varphi)}{\sin \frac{1}{2}\,(\delta + \varphi)} \quad . \quad . \quad . \quad (207)$$

from which A becomes known. The angle ξ, or that at the body is technically called the *angle of variation.*

§ 688. The north star, called *Polaris*, is often advantageously used for this purpose, particularly when it has its greatest eastern or western elongation. At that time the vertical circle through the star is tangent to its diurnal path, and its diurnal motion will be in altitude alone, and not at all in azimuth, thus affording time for taking a series of azimuthal distances.

Fig. 109.



To find the time when Polaris or other circumpolar star has its greatest elongation, observe that the angle of variation is at that instant 90°, and in the right-angled triangle $P\,Z\,S$, right-angled at S, we have

$$\cos P = \tan \delta \,.\, \cot \varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (208)$$

and

$$T_{*} = A_{*} + \tfrac{1}{15} \,.\, \cos^{-1} [\tan \delta \,.\, \cot \varphi] \,.\quad . \quad . \quad . \quad (209)$$

Also

$$\sin A = \frac{\sin \delta}{\sin \varphi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (210)$$

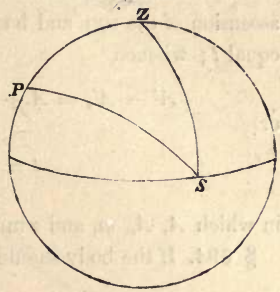
§ 689. When the bearing of the sun is taken, the line of collimation must be directed to one or the other extremity of his horizontal diameter; to the bearing of which must be added the horizontal semi-diameter reduced to the horizon, which is equal to the tabular semi-diameter divided by the cosine of the sun's altitude.

§ 690. *Variation of the Compass.*—From the foregoing it will be easy to find the variation of the compass, or, as it is frequently called, the *declination of the magnetic needle.* For this purpose it will be sufficient to take the magnetic bearing of some heavenly body and note the time. Then from the time and equations (206) and (207), computing the true azimuth, and taking the difference between the magnetic and true azimuths, the problem is solved.

Or, if the true bearing of any terrestrial object be known, we have only to subtract it from the magnetic bearing, as determined by the compass, to obtain the same result.

§ 691. At sea, or on land where the observer is surrounded by prairies or extended plains, it is usual to take the magnetic bearing of the sun's centre, by observing alternately the opposite horizontal limbs at the time of rising. Then in the triangle $Z S P$, we have

Fig. 110.



$$Z S = 90° + \text{refraction} - \text{parallax} = \zeta;$$

and making

$$2\psi = \zeta + \delta + \varphi,$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin \psi \, . \sin (\psi - \delta)}{\sin \zeta \, . \sin \varphi}} \quad . \quad . \quad . \quad . \quad (211)$$

It will be sufficient to regard the horizontal refraction and parallax as constant, and the former equal to $33' \, 45''$ and the latter to $8''$; thus making

$$\zeta = 90° \; 33' \; 37''.$$

MERIDIAN PASSAGE.

§ 692. It is often desirable to know in advance what will be the indication of a sidereal or mean solar time-piece at the instant a given body is on the meridian. This indication will measure the hour angle of the vernal equinox, or of the mean sun at the instant, according as the timekeeper is running to sidereal or mean solar time.

§ 693. *Time of Meridian Passage.*—To find the local mean solar time of a given body coming to the meridian, make

$t =$ the time required;

$A'_s =$ right ascension of mean sun at this time;

$A' =$ right ascension of the body at the same instant;

$A_s =$ right ascension of mean sun at Greenwich, mean noon next previous;

$A =$ right ascension of the body at the same instant;

$s =$ hourly change of mean sun in right ascension;

$m =$ hourly change of body in right ascension;

$l =$ longitude of place in time.

Then the time at Greenwich corresponding to the local time t, will be $t + l$; and

$$A'_s = A_s + s \;\; (t + l),$$
$$A' = A \;\, + m \, (t + l);$$

and since all the elements are expressed in time, the difference of right ascension of the sun and body, when the latter is on the meridian, must equal t; whence

$$A' - A'_s = A + m\,(t + l) - A_s - s\,(t + l) = t;$$

or

$$t = \frac{A - A_s + l\,(m - s)}{1 - (m - s)} \quad . \quad . \quad . \quad . \quad . \quad (212)$$

in which A, A_s, m, and s must be expressed in the same unit, say hours.

§ 694. If the body should be a star, then will $m = 0$, and

$$t = \frac{A - A_s - l\,s}{1 + s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (213)$$

695. If a planet with retrograde motion, m would change its sign, and

$$t = \frac{A - A_s - l\,(m + s)}{1 + m + s} \quad . \quad . \quad . \quad . \quad . \quad (214)$$

§ 696. If the sidereal time were asked for, then would $A_s = 0$, $s = 0$, and

$$t = \frac{A + l\,m}{1 - m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (215)$$

and if the body be a star, then $m = 0$, and

$$t = A.$$

REDUCTION TO THE MERIDIAN.

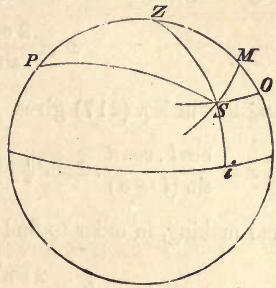
§ 697. Some of the most important astronomical determinations depend upon the measured zenith distances or altitudes of a body when on the meridian; but these measurements it is not always convenient nor possible to make, and besides it is desirable to multiply measurements as much as possible to secure the advantages of a general average in eliminating errors of observations. The purpose of the next proposition is, therefore, to pass from a measured zenith distance or altitude taken when the body is off the meridian to what it would have been had the body been on that circle.

The difference between any two zenith distances, applied with the proper sign to either, will give the other; and when one is the meridian zenith distance, this difference is called the *reduction to the meridian.*

§ 698. *Reduction to the Meridian.*—To find the reduction to the meridian.

Let P be the pole, Z the zenith, S a star, $S\,M$ an arc of the star's diurnal circle cutting the meridian in M, $S\,O$ the arc of a horizontal circle through the star, and cutting the meridian in O. Make

Fig. 111.



$x = Z\,S - Z\,M = Z\,O - Z\,M =$ reduction to meridian;

$l =$ latitude of place;

$d =$ declination of star;

$P =$ hour angle $Z\,P\,S$;

$z =$ zenith distance $Z\,S$.

Then because

$$P\,Z = 90° - l;\ P\,S = 90° - d;$$

we have in the triangle $P\,Z\,S$

$$\cos z = \sin l \,.\, \sin d + \cos l \,.\, \cos d \,.\, \cos P;$$

but

$$\cos P = 1 - 2 \,.\, \sin^2 \tfrac{1}{2} P;$$

and substituting this we get

$$\begin{aligned}\cos z &= \sin l \,.\, \sin d + \cos l \,.\, \cos d - 2 \cos l \,.\, \cos d \sin^2 \tfrac{1}{2} P, \\ &= \cos (l - d) - 2 \cos l \,.\, \cos d \,.\, \sin^2 \tfrac{1}{2} P.\end{aligned}$$

But $Z\,M = l - d$; and $z = x + l - d$; and therefore,

$$\cos z = \cos x \,.\, \cos (l - d) - \sin x \sin (l - d);$$

also,

$$\cos x = 1 - \tfrac{1}{2} x^2 +, \&c.;$$

and if the observations be made near the meridian, x will be very small, and its powers higher than the second may be neglected. Making this supposition, writing the arc for its sine, and substituting the value of $\cos x$ above, we have

$$\cos z = (1 - \tfrac{1}{2} x^2) \,.\, \cos (l - d) - x \,.\, \sin (l - d).$$

Equating these values of $\cos z$, there will result

$$\tfrac{1}{2} x^2 \,.\, \cos (l - d) + x \sin (l - d) = 2 \cos l \,.\, \cos d \,.\, \sin^2 \tfrac{1}{2} P \quad . \quad . \quad (216)$$

In consequence of the small value of x, it will be sufficient for all practical purposes to make an approximate solution of this equation; for this purpose write it

$$x = \frac{2 \cos l \,.\, \cos d}{\sin (l - d)} \,.\, \sin^2 \tfrac{1}{2} P - \text{cotan} (l - d) \,.\, \tfrac{1}{2} x^2 \quad . \quad . \quad (217)$$

neglecting the term involving the second power of x,

$$x = \frac{2 \cos l \,.\, \cos d}{\sin (l - d)} \,.\, \sin^2 \tfrac{1}{2} P \quad . \quad . \quad . \quad . \quad . \quad . \quad (218)$$

and this in Eq. (217) gives

$$x = \frac{\cos l \,.\, \cos d}{\sin (l - d)} \,.\, 2 \sin^2 \tfrac{1}{2} P - \cot (l - d) \,.\, \left(\frac{\cos l \,.\, \cos d}{\sin (l - d)}\right)^2 \,.\, 2 \sin^4 \tfrac{1}{2} P,$$

and making, in order to find x in seconds of arc,

$$k = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}; \quad m = \frac{2 \sin^4 \frac{1}{2} P}{\sin 1''} \quad . \quad . \quad . \quad . \quad . \quad (219)$$

$$x = k \,.\, \frac{\cos l \,.\, \cos d}{\sin (l - d)} - m \,.\, \cot (l - d) \,.\, \left(\frac{\cos l \,.\, \cos d}{\sin (l - d)}\right)^2 \quad . \quad (220)$$

§ 699. Now P is to be found from the time when the star is on the meridian and that of observation, being equal to the difference of the two converted into arc. These times are to be taken from a time-piece, and this never runs accurately to sidereal or mean solar time. If the time-keeper run too slow, the difference of its indications would be less than the corresponding difference of true hour angles—if too fast, the contrary; and P, in the formula, must be corrected.

Let the time-piece lose r seconds a day; then while the true day will be equal to 86400^s, the clock indication will be $86400^s - r$, and any two corresponding hour angles, one being the true and the other that indicated by the time-keeper, denoted respectively by P and P', will bear the relation

$$P : P' :: 86400 : 86400 - r;$$

whence

$$P = P' \,.\, \frac{86400}{86400 - r} = P' \,.\, \frac{1}{1 - \dfrac{r}{86400}};$$

making

$$r' = \frac{r}{86400} = 0.000011 \,.\, r \quad . \quad . \quad . \quad . \quad . \quad . \quad (221)$$

developing the fraction, and neglecting the higher powers of r',

$$P = P' (1 + r') = P' + P' r',$$

and

$$\sin \tfrac{1}{2} P = \sin \tfrac{1}{2} P' \cos \tfrac{1}{2} r' P' + \cos \tfrac{1}{2} P' \sin \tfrac{1}{2} r' P';$$

making $\cos \frac{1}{2} r' P' = 1$, squaring and rejecting the term containing the second power of $\sin \frac{1}{2} r' P'$, we find

$$\sin^2 \tfrac{1}{2} P = \sin^2 \tfrac{1}{2} P' + 2 \sin \tfrac{1}{2} P' \cos \tfrac{1}{2} P' \,.\, \sin \tfrac{1}{2} r' P' \,;$$

but

$$2 \sin \tfrac{1}{2} P' \,.\, \cos \tfrac{1}{2} P' = \sin P',$$

and since P' and r' are both small,

$$\sin P' = 2 \sin \tfrac{1}{2} P',$$
$$\sin \tfrac{1}{2} r' P' = r' \sin \tfrac{1}{2} P' \,;$$

which substituted above give

$$\sin^2 \tfrac{1}{2} P = \sin^2 \tfrac{1}{2} P' + 2\, r' \sin^2 \tfrac{1}{2} P' = (1 + 2\, r') \sin^2 \tfrac{1}{2} P' \,;$$

and finally making

$$i = 1 + 2\, r' = 1 + 0.000022\, r \quad . \quad . \quad . \quad . \quad (222)$$

and substituting in Eq. (220) we have

$$x = i \,.\, k \,.\, \frac{\cos l \,.\, \cos d}{\sin (l - d)} - i^2 \,.\, m \,.\, \cot (l - d) \,.\, \left(\frac{\cos l \,.\, \cos d}{\sin (l - d)}\right)^2 \quad (223)$$

in which it will be recollected that r, in the value of i, is the rate of the time-keeper, minus when the latter gains and plus when it loses on sidereal time.

§ 700. The first term in the second member of Eq. (223) will always be sufficient when the observations are made within five or ten minutes of the meridian. And it is important to remark, in view of the use presently to be made of the value of x, that the latter will not be sensibly affected by a small error in the value of l, and that an approximate latitude may therefore be substituted therefor. The values of k and m are computed for all values of P' from 0 to 35^m, and inserted in Tables V. and VI.

TERRESTRIAL LATITUDE AND LONGITUDE.

§ 701. The determinations of terrestrial latitude and longitude by means of astronomical observations and ephemerides, are among the most important of the objects of practical astronomy. All appreciate the value of these determinations in navigation and geography, and we now proceed to consider them in the order named.

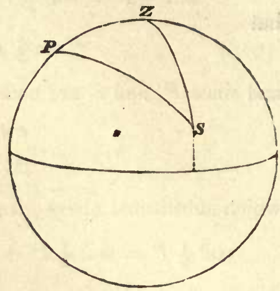
Terrestrial Latitude.

§ 702. The zenith distance of the pole is always the complement of the latitude of the place, and when known the latitude is known from the relation

$$\lambda = 90^\circ - l,$$

in which λ denotes the zenith distance of the pole, and l the latitude of the place.

Fig. 112.



§ 703. The zenith distance of the pole forms one side ZP of a spherical triangle, of which the two other sides, ZS and PS, form, respectively, the zenith and polar distances of some heavenly body, of which the angle at the pole is the hour angle, or distance of the body from the meridian. And the determination of latitude consists in the solution of this triangle, the data for this purpose being the true zenith distance ZS determined from observation, the polar distance PS found from the ephemeris, and the hour angle ZPS, which is always equal to the sidereal time of observation, diminished by the body's right ascension at the same instant. Having, then, found the true zenith distance by correcting the observed for refraction, parallax, and semi-diameter when necessary, and the body's true hour angle and polar distance from the time of observation, the ordinary formulas for the solution of spherical triangles will do the rest.

§ 704. *Latitude by Meridian Zenith Distance of a Body.*—But it is desirable, in practice, to select those moments for observations which will give most accurate results, and these are when the hour angle is 0° or 180°; in other words, when the body is on or near the meridian, for then it has the least change in zenith distance for a given interval of time.

Make

$$z = ZS \quad = \text{true zenith distance of body;}$$
$$d = 90^\circ - PS = \text{the body's declination;}$$
$$P = ZPS \quad = \text{hour angle of the body;}$$
$$A = PZS = 180^\circ - \text{the body's azimuthal angle.}$$

Then in the triangle ZPS,

$$\cos z = \sin l \,.\, \sin d + \cos l \,.\, \cos d \,.\, \cos P \quad . \quad . \quad . \quad (224)$$

$$\sin d = \sin l \,.\, \cos z + \cos l \,.\, \sin z \,.\, \cos A \quad . \quad . \quad . \quad (225)$$

§ 705. Making $P = 0^\circ$, the body will be on the meridian somewhere between the poles on the side of the zenith, and A will be 0° or 180°.

In the first case, the body will be between the zenith and elevated pole $\cos A = 1$, and Eq. (225) will become

$$\sin d = \sin l \,.\, \cos z + \cos l \,.\, \sin z = \sin (l + z),$$

whence

$$d = l + z,$$

and

$$l = d - z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (226)$$

Fig. 113.

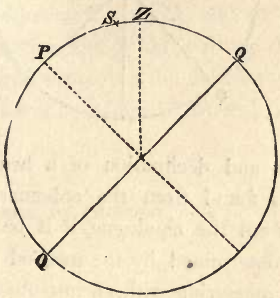
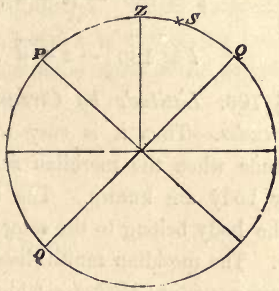


Fig. 114.



In the second case, the body will be on the opposite side of the zenith from the elevated pole, $\cos A = -1$; and if the latitude and declination be of the same name, $\sin d$ and $\sin l$ will have the same sign, and Eq. (225) gives

$$\sin d = \sin l \,.\, \cos z - \cos l \,.\, \sin z = \sin (l - z);$$

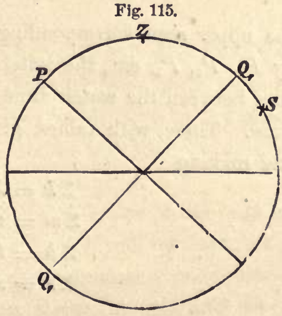
whence

$$d = l - z,$$

and

$$l = d + z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (227)$$

Fig. 115.



If, in the second case, the declination and latitude be not of same name, the body will be below the equinoctial; $\sin d$ and $\sin l$ will have contrary signs, and Eq. (225) gives

$$\sin (-d) = \sin l \,.\, \cos z - \cos l \,.\, \sin z = \sin (l - z);$$

whence

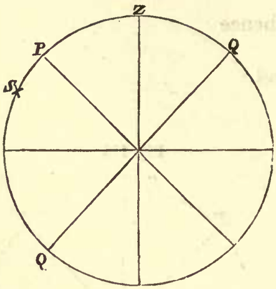
$$-d = l - z,$$

and

$$l = z - d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (228)$$

If $P = 180°$, the body will be on the meridian below the elevated pole, and $A = 0°$; $\cos P = -1$, and, Eq. (224),

Fig. 116.



$$\cos z = \sin l . \sin d - \cos l . \cos d = -\cos (l + d);$$

whence

$$z = 180° - (l + d),$$

and

$$l = 180° - \overline{z + d} \quad . \quad . \quad (229)$$

§ 706. *Latitude by Circum-meridian Altitudes.*—Thus it is easy to find the latitude when the meridian zenith distance and declination of a heavenly body are known. The declination is found from the ephemeris, if the body belong to the solar system, or from the catalogue, if it be a star. The meridian zenith distance is best determined by the method of *circum-meridian altitudes,* which consists in measuring with an instrument a number of altitudes of the body just before and after its meridian passage, noting the corresponding times; reducing to the meridian, taking an average value of the results, and subtracting this from 90°.

§ 707. Denote by h_1, h_2, h_3, &c., the measured altitudes; r_1, r_2, r_3, &c., the corresponding refractions; p_1, p_2, p_3, &c., the parallaxes; Δ the apparent semi-diameter; x_1, x_2, x_3, &c., the reductions to the meridian; n the number of observations; and H the average meridian altitude; then will

$$H = \frac{h_1 - r_1 + p_1 + x_1 + h_2 - r_2 + p_2 + x_2 + \text{\&c.}}{n} \pm \Delta \quad . \quad (230)$$

the upper sign corresponding to the lower limb, and *vice versa.* Denote by P_1, P_2, P_3, &c., the watch hour angles of the body; that is, the difference between the watch time of meridian passage and those of observations. These, with tables, give k_1, k_2, k_3, &c., m_1, m_2, m_3, &c., Eq. (223); and making

$$\Sigma k = k_1 + k_2 + k_3 +, \text{\&c.};$$
$$\Sigma m = m_1 + m_2 + m_3 +, \text{\&c.};$$
$$\Sigma h = h_1 + h_2 + h_3 +, \text{\&c.};$$
$$\Sigma x = x_1 + x_2 + x_3 +, \text{\&c.};$$

$$H = \frac{\Sigma h}{n} - \frac{\Sigma r}{n} + \frac{\Sigma p}{n} + i . \frac{\Sigma k}{n} . \frac{\cos l . \cos d}{\sin (l - d)} - i^2 . \frac{\Sigma m}{n} . \cot (l - d) \left(\frac{\cos l . \cos d}{\sin (l - d)}\right)^2 \pm \Delta$$

(231)

But this supposes l to be known. An approximate value will, § 700, be sufficient; and to obtain it, correct the altitude nearest the meridian for

refraction, parallax, and semi-diameter; subtract the result from 90°, and substitute the remainder for z in one of the equations (226) to (229) inclusive, according to the case.

§ 708. *Latitude by opposite and nearly equal Meridian Zenith Distances.*—With an approximate latitude, select one or more pairs of stars, of which the individuals of each pair shall pass the meridian on opposite sides of the zenith, and at nearly equal distances. Then, preserving the notation of § 704, writing the subscripts 1 and 2 to distinguish the stars, and supposing the declinations to be of the same name as the latitude, we have, equations (226) and (227),

$$l = d_1 + z_1,$$
$$l = d_2 - z_2;$$

and, by addition,

$$l = \frac{d_1 + d_2}{2} + \frac{z_1 - z_2}{2}.$$

Denoting by ζ_1 and ζ_2 the observed zenith distances, and by r_1 and r_2 the corresponding refractions, we have

$$z_1 = \zeta_1 + r_1, \quad z_2 = \zeta_2 + r_2;$$

which, substituted above, give

$$l = \frac{d_1 + d_2}{2} + \frac{\zeta_1 - \zeta_2}{2} + \frac{r_1 - r_2}{2} \quad . \quad . \quad . \quad . \quad (232)$$

If $\zeta_1 = \zeta_2$, then will $r_1 - r_2 = 0$, and we have

$$l = \frac{d_1 + d_2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (233)$$

and thus the determination of latitude will be made independent of refraction, which is one of the greatest sources of difficulty in practical astronomy.

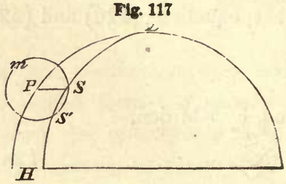
If ζ_1 be not equal to ζ_2, but nearly so, the result may be regarded as equally accurate, since the difference of refraction will then be employed, which, being very small, will be sensibly free from error.

§ 709. This simple and elegant method, which is one of the most accurate, and now very generally used, was first employed by Capt. Andrew Talcott, late of the U. S. Engineers. The measurements were made by means of a zenith telescope, turning about a vertical axis, and provided with a micrometer. The stars were so selected that when one was brought within the field of view, and made to thread the micrometer wire as it passed the meridian, the other would enter the field on turning the instrument 180° in azimuth. The second star being made to thread the wire

by the micrometer motion, the extent of the latter was noted, and gave the value of $\zeta_1 - \zeta_2$. The value of $r_1 - r_2$ was found, of course, from the refraction tables.

§ 710. *Latitude by Polaris off the Meridian.*—The last method we shall give is that by Professor Littrow. It consists in observing the altitude of *Polaris* out of the meridian, and therefore at any convenient time, and reducing, not to the meridian only, but to the pole also; the data for this purpose being the star's polar distance, its true altitude, and corresponding hour angle.

Fig. 117



Let Z be the zenith, P the pole, S the place of the star in its diurnal path $S\,S'\,m$, $Z\,S$ the arc of a vertical circle. Make

$l =$ latitude $=$ altitude of pole $= 90^\circ - Z\,P$;

$h =$ true altitude of star $= 90^\circ - Z\,S$;

$P = Z\,P\,S =$ hour angle of star;

$\psi =$ reduction to the pole $= Z\,P - Z\,S$;

$\Delta = P\,S =$ star's polar distance.

Then

$$\psi = h - l;$$

and

$$l = h - \psi;$$

so that the latitude is known when ψ is known.

In the triangle $Z\,P\,S$, we have

$$\cos Z\,S = \cos P\,S \,.\, \cos Z\,P + \sin P\,S \,.\, \sin Z\,P \,.\, \cos P;$$

and replacing the sides by their values in terms of Δ, h, and l, or $h - \psi$.

$$\sin h = \cos \Delta \,.\, \sin (h - \psi) + \sin \Delta \,.\, \cos (h - \psi) \,.\, \cos P,$$

dividing by $\sin h$ and factoring,

$$1 = \cos \psi \,.\, (\cos \Delta + \sin \Delta \cot h \,.\, \cos P) - \sin \psi \,.\, (\cos \Delta \,.\, \cot h - \sin \Delta \,.\, \cos P).$$

Make

$$\left.\begin{aligned} a &= \cos \Delta + \sin \Delta \,.\, \cot h \,.\, \cos P, \\ b &= \cos \Delta \,.\, \cot h - \sin \Delta \,.\, \cos P; \end{aligned}\right\} \quad . \; . \; . \; . \quad (234)$$

and the above may be written,

$$1 = a \cos \psi - b \,.\, \sin \psi \quad . \; . \; . \; . \; . \; . \; . \quad (235)$$

Now, Δ is a small angle, not more than $1^\circ\ 40'$; and replacing $\cos \Delta$ and

sin Δ by their values in terms of Δ, equations (234) become, omitting the powers of Δ above the third,

$$a = 1 - \tfrac{1}{2}\Delta^2 + (\Delta - \tfrac{1}{6}\Delta^3) \,.\, \cot h \,.\, \cos P,$$
$$b = (1 - \tfrac{1}{2}\Delta^2) \,.\, \cot h - (\Delta - \tfrac{1}{6}\Delta^3) \cos P.$$

Let

$$\psi = A\Delta + B\Delta^2 + C\Delta^3 +, \&c. \quad . \quad . \quad . \quad . \qquad (236)$$

be the development of ψ according to the ascending powers of Δ, in which there can be no independent term; since, when $\Delta = 0$, then will $\psi = 0$. Whence

$$\cos \psi = 1 - \tfrac{1}{2}A^2\Delta^2 - AB\Delta^3,$$
$$\sin \psi = A\Delta + B\Delta^2 + (C - \tfrac{1}{6}A^3)\Delta^3.$$

Substituting the values of a, b, $\cos \psi$, and $\sin \psi$, in Eq. (235), we have the identical equations,

$$\cot h \,.\, \cos P - A \,.\, \cot h = 0,$$
$$-\tfrac{1}{2}(1 + A^2) + A \cos P - B \cot h = 0,$$
$$\tfrac{1}{2}A - \tfrac{1}{6}(1 + 3A^2)\cos P - (C - \tfrac{1}{6}A^3) = 0.$$

Whence

$$A = \cos P;$$
$$B = -\tfrac{1}{2}\sin^2 P \,.\, \tan h;$$
$$C = \tfrac{1}{3}\cos P \,.\, \sin^2 P;$$

which in Eq. (236) give

$$\psi = \Delta \,.\, \cos P - \tfrac{1}{2}\sin^2 P \,.\, \tan h \,.\, \Delta^2 + \tfrac{1}{3}\cos P \,.\, \sin^2 P \,.\, \Delta^3.$$

To express ψ and Δ in seconds, write $\psi \sin 1''$ for ψ and $\Delta \sin 1''$ for Δ, and make

$$m = \tfrac{1}{2}\sin 1'', \; n = \tfrac{1}{3}\sin^2 1'',$$

then will

$$\psi = \Delta \cos P - m\,(\Delta \,.\, \sin P)^2 \,.\, \tan h + n \,.\, (\Delta \,.\, \cos P) \,.\, (\Delta \,.\, \sin P)^2 \quad (237)$$

This value applied with its proper sign to the observed altitude, corrected for refraction, will give the latitude. It is best to take some half dozen altitudes, and to note the corresponding times in pretty rapid succession; a mean of the altitudes corrected for refraction will give h, and a mean of the sidereal times diminished by the right ascension of the star, and the remainder multiplied by 15, will give P.

§ 711. This method is of such practical utility as to have caused the insertion into the English Astronomical Ephemeris and Nautical Almanac of three tables, of which the first contains the value of $\Delta \cos P$ for every 10 minutes, sidereal time, for a mean and constant value of Δ; the second contains the values of $-m \,.\, (\Delta \,.\, \sin P)^2 \,.\, \tan h$; and the third contains

corrections to be applied to the values in the second table. The second and third tables are arranged in the form of double entry, the arguments for the former being the sidereal time and altitude, and in the latter sidereal time and date.

The third term of Eq. (237) is neglected as being insignificant.

Longitude.

§ 712. The longitude of a place is the angle made by its meridian with some assumed meridian taken as an origin of reference. The problem of longitude is much more complex than that of latitude, and its solution consists, as we have seen, § 94, in finding the difference of local times that exist simultaneously on the required and first meridian.

§ 713. *Longitude by Chronometers.*—Could the motion of a time-piece be made perfectly uniform, and the angular velocity of its hour-hand equal to that of the earth's axial rotation, without the risk of variation, the determination of longitude would be a simple matter. It would then only be necessary to put the time-keeper in motion; on a given meridian ascertain, by the methods explained, its error on the local time of this meridian; transport it to the unknown meridian, determine its error on local time there, and take the difference of these errors; this difference would be the difference of longitude of the meridians in time.

But such time-pieces cannot be made. The results to which they would lead may, however, be approached within limits all-sufficient for practical purposes. It is only necessary that the time-keeper shall run uniformly, a condition which chronometers have been made so nearly to attain as to vary their rate but half a second in 31536000 seconds.

§ 714. By daily observations find the error of a chronometer; from the variation of the error during the intervals between the observations, find that for 24 chronometer hours. This will be the *rate*. Make

$e =$ error on local time on given meridian, at some given epoch; plus when too slow, minus when too fast;

$e_{\prime} =$ error on local time on required meridian, at some subsequent epoch;

$e_{\prime\prime} =$ error on local time on given meridian, at this last epoch;

$r =$ rate; minus when gaining, plus when losing;

$i =$ interval of *chronometer time* between the epochs at which e and $e_{\prime}$ are found—always plus;

$l =$ difference of longitude.

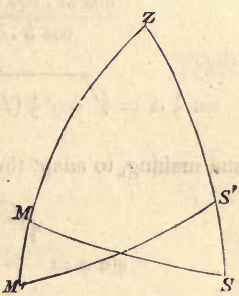
Then
$$l = e_{\prime} - e_{\prime\prime};$$
$$e_{\prime\prime} = e + ir;$$

whence $$l = e_{,} - \overline{e + i.r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (238)$$

§ 715. *Longitude by Lunar Distances.*—The moon has a rapid motion in longitude. Her geocentric angular distances from the sun, planets, and fixed stars that lie in and about her path through the heavens, are computed in advance and inserted into the Nautical Almanac. From these hours and distances is readily found, by interpolation, the Greenwich time corresponding to any given distance not in the Almanac, and the difference between this interpolated time and the local time on any other meridian at which the moon is found from observation to have this given distance, is the longitude of the meridian on which the observation is made.

§ 716. Measure the altitude of the star, and that of the upper or lower bright limb of the moon; also measure the angular distance from the star to the bright limb of the moon, and note the local time of this measurement; correct the altitude of the limb and measured distance for semi-diameter; then correct the altitude of the star for refraction, and that of the moon for refraction and parallax.

Fig. 118.



Let Z be the zenith, $Z\,S$ and $Z\,M$ the arcs of vertical circles, the first passing through the star S and the second through the moon's centre M. The effect of refraction being to elevate and that of parallax to depress, and the parallax of the moon being always greater than her refraction, the star will appear at S' above its true place, and the moon at M' below her true place.

Make

$h = 90° - Z\,M'$ = observed altitude of moon's limb corrected for semi-diameter;

$h' = 90° - Z\,S'$ = observed altitude of star;

$\Delta' = M'S'$ = observed distance corrected for semi-diameter of the moon;

$H = 90° - Z\,M$ = true altitude of moon's centre;

$H' = 90° - Z\,S$ = true altitude of star;

$\Delta = M\,S$ = true or geocentric distance between the moon's centre and the star;

$z = M\,Z\,S$ = angle at Z.

Then in the triangle $M'Z\,S'$,

$$\cos z = \frac{\cos \Delta' - \sin h \,.\, \sin h'}{\cos h \,.\, \cos h'},$$

and in triangle $M\,Z\,S$,

$$\cos z = \frac{\cos \Delta - \sin H . \sin H'}{\cos H . \cos H'};$$

equating these values of $\cos z$,

$$\frac{\cos \Delta' - \sin h . \sin h'}{\cos h . \cos h'} = \frac{\cos \Delta - \sin H . \sin H'}{\cos H . \cos H'};$$

adding unity to both members and reducing,

$$\frac{\cos \Delta' + \cos (h + h')}{\cos h . \cos h'} = \frac{\cos \Delta + \cos (H + H')}{\cos H . \cos H'}.$$

Make

$$h + h' + \Delta' = 2\,m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (239)$$

whence

$$\cos (h + h') = \cos (2\,m - \Delta');$$

substituting this above and reducing, we find

$$\frac{\cos m . \cos (m - \Delta')}{\cos h . \cos h'} = \frac{\cos^2 . \frac{H + H'}{2} - \sin^2 \frac{\Delta}{2}}{\cos H . \cos H'},$$

whence

$$\sin \tfrac{1}{2} \Delta = \sqrt{\cos^2 \tfrac{1}{2} (H + H') - \frac{\cos H . \cos H'}{\cos h . \cos h'} . \cos m . \cos (m - \Delta')},$$

and making, to adapt the foregoing to logarithmic computation,

$$\sin \varphi = \frac{\sqrt{\frac{\cos H . \cos H'}{\cos h . \cos h'} . \cos m . \cos (m - \Delta')}}{\cos \frac{1}{2} (H + H')} \quad . \quad . \quad (240)$$

then will result

$$\sin \tfrac{1}{2} \Delta = \cos \tfrac{1}{2} (H + H') . \cos \varphi \quad . \quad . \quad . \quad . \quad (241)$$

§ 717. The quantities h, h', H, H', and Δ', are obtained from observations, and the corrections for semi-diameter, refraction, and parallax applied thereto; the value of m is given by Eq. (239); the auxiliary arc φ by Eq. (240), and, finally, the true distance Δ by Eq. (241.) This operation is technically called *clearing the distance.*

§ 718. With this distance enter the Nautical Almanac and see if it is found therein; if it is, take the corresponding time from the head of the column, and subtract therefrom the local time of observation; the remainder will be the longitude—west if this remainder be plus, east if it be negative.

§ 719. If the precise distance be not found in the Almanac, as it sel

dom will, find two consecutive distances, one of which is greater and the other less. Take these and the next two or more preceding and following distances, and form their first, second, third, &c., differences, denoted respectively by Δ_1, Δ_2, Δ_3, &c., in which Δ_1 is the difference between the consecutive distances of which one is less and the other greater than the given distance. Make

$D =$ given distance;
$D' =$ nearest distance in ephemeris;
$T' =$ ephemeris time corresponding to D',
$T =$ Greenwich time corresponding to D:
$t = T - T'$.

Then because the ephemeris intervals are 3^h, will, by the ordinary formula for interpolation,

$$D = D' + \frac{t}{3^h} \cdot \Delta_1 + \frac{t\,(t - 3^h)}{(3^h)^2 . 2} \Delta_2 + \&c;$$

supposing the second differences constant, which we may do without sensible error, and solving with respect to first power of t,

$$t = \frac{D - D'}{\Delta_1 - \frac{1}{2}\Delta_2 + \frac{1}{2}\Delta_2 \cdot \frac{t}{3^h}} 3^h \quad . \; . \; . \; . \; . \; . \; . \quad (242)$$

Neglecting the second difference, we have

$$t = \frac{D - D'}{\Delta_1} 3^h \quad . \; . \; . \; . \; . \; . \; . \; . \quad (243)$$

which in the denominator of the preceding equation gives

$$t = \frac{D - D'}{\Delta_1 - \frac{1}{2}\Delta_2 + \frac{1}{2}(D - D') \cdot \frac{\Delta_2}{\Delta_1}} 3^h . \; . \; . \; . \quad (244)$$

and replacing t by its value $T - T'$, we have finally

$$T = T' + \frac{D - D'}{\Delta_1 - \frac{1}{2}\Delta_2 + \frac{1}{2}(D - D') \cdot \frac{\Delta_2}{\Delta_1}} 3^h . \; . \; . \quad (245)$$

§ 720. A single observer begins by taking with his sextant an altitude of the star, then an altitude of the moon's bright limb, then the distance between the star and moon's limb, then the altitude of the moon's bright limb, then the altitude of star, carefully noting the time of taking the distance. A mean of the altitudes of the moon and star will give the approximate altitudes which the moon and star had when the distance was measured.

§ 721. It is scarcely necessary to add, that if the sun or a planet be taken instead of a star, corrections for semi-diameter and parallax must be added to that of refraction.

§ 722. *Longitude by Lunar Culminations.*—If the change in right ascension of a point of the moon, in its passage from one meridian to another, be known, the distance between the meridians becomes known from the point's *rate* of motion in right ascension. Make

c_1 = the point's right ascension when on any upper first meridian;

c_2 = its right ascension when on an upper known meridian to the west.

H = longitude of this known meridian, west.

l = approximate longitude of any unknown meridian between these.

L = true longitude of the same.

$e = L - l$.

α = right ascension of the point when on the upper meridian, of which the longitude is L.

§ 723.—1*st Approximation.* Then, were point's motion in right ascension uniform,

$$c_2 - c_1 : H :: \alpha - c_1 : l$$

or

$$l = \frac{H}{c_2 - c_1} (\alpha - c_1)$$

§ 724.—2*d Approximation.* But the moon's motion in right ascension is not uniform, and the above will in general be erroneous, and by the quantity e, which is a small arc of longitude; and we have

$$L = l + e$$

The arc e being small, the moon's motion in right ascension will be sensibly uniform while between the meridians, through its extremities. Make

α_1 = the lunar point's right ascension when on the meridian, of which the longitude is l,

v = the point's rate of motion in right ascension; and let this be measured by the distance in right ascension over which the point would move, with this rate constant, while between the meridians of which the distance apart is H.

Then by the principle above, writing e for l, v for $c_2 - c_1$, and α_1 for c_1, we have

$$e = \frac{H}{v} \cdot (\alpha - \alpha_1);$$

and this in the above gives

$$L = l + \frac{H}{v} \cdot (\alpha - \alpha_1)$$

§ 725. Now, these equations will be equally true from whatever point of the equinoctial, taken as an origin, the right ascension be estimated. For convenience, take the origin at the declination circle through the lunar point at its last passage over the first, or meridian of the Ephemeris. Then will

$c_1 = 0,$

$c_2 =$ change of right ascension between the known meridians,

$\alpha =$ increase of right ascension from the first to the intermediate or required meridian,

$\alpha_1 =$ increase of right ascension from the first to the approximate meridian l.

With this new notation the above equations become

$$l = \frac{H}{c_2} \cdot \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (246)$$

$$L = l + \frac{H}{v} \cdot (\alpha - \alpha_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (247)$$

§ 726. In the Nautical Almanac and Astronomical Ephemeris are given the right ascension of the point of the bright limb at which a declination circle is tangent to the lunar disc, and also the right ascensions of one or more stars, at the instant of passing the upper and lower meridian of Greenwich for every day in the year. The stars are so situated as to lie about the moon's parallel of declination, and not far from her in right ascension.

§ 727.—1. *Interpolation.* Take the following scheme:

I	F	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
t'''	a'''					
		b''				
t''	a''		c''			
		b'		d'		
t'	a'		c'		e'	
		b		d		f
$t_{,}$	$a_{,}$		$c_{,}$		$e_{,}$	
		$b_{,}$		$d_{,}$		
$t_{,,}$	$a_{,,}$		$c_{,,}$			
		$b_{,,}$				
$t_{,,,}$	$a_{,,,}$					

in which the column I contains the independent variable, or argument, as time, terrestrial longitude, degrees, and the like; F the value of a

function of this variable, as found in any set of tables; Δ_1, Δ_2, Δ_3, etc., the first, second, third, etc., orders of differences of these functions.

Make

s = the interpolated value of the function corresponding to any given value t_s of the argument between t' and $t_,$;

$$\left.\begin{aligned} t &= \frac{t_s - t'}{t_, - t'}; \\ \Delta_1 &= b, \\ \Delta_2 &= \frac{c' + c_,}{2}, \\ \Delta_3 &= d, \\ \Delta_4 &= \frac{e' + e_,}{2}, \\ \Delta_5 &= f. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (248)$$

Then, limiting the operation to the fourth order of differences, will

$$s = a' + At + Bt^2 + Ct^3 + Dt^4 +$$
$$\alpha_1 = s - a' = At + Bt^2 + Ct^3 + Dt^4 \qquad (249)$$

in which

$$\left.\begin{aligned} A &= \Delta_1 - \tfrac{1}{2}\Delta_2 + \tfrac{1}{12}\Delta_3 + \tfrac{1}{12}\Delta_4, \\ B &= \tfrac{1}{2}\Delta_2 - \tfrac{1}{4}\Delta_3 - \tfrac{1}{24}\Delta_4, \\ C &= \tfrac{1}{6}\Delta_3 - \tfrac{1}{12}\Delta_4, \\ D &= \tfrac{1}{24}\Delta_4, \end{aligned}\right\} \quad \ldots \quad (250)$$

Also taking first differential coefficient of the function (249)

$$v = A + 2Bt + 3Ct^2 + 4Dt^3 \quad \ldots\ldots \quad (251)$$

which would be the increment of the function for an increment of t equal to unity, were the function to increase uniformly and at the rate it had for any arbitrary value for t_s.

§ 728.—2. *Observations.* Make

$s_☽$ = sidereal time of moon's bright limb passing meridian.
$h_☽$ = clock time of same passing line of colimation.
$e_☽$ = clock error at same instant.
$i_☽$ = error of transit for altitude of moon, in time seconds.

Then, §§ 211 and 729,

$$s_{☽} = h_{☽} + e_{☽} \pm \frac{i_{☽}}{1 - 0{,}04166\,.\,m}(1 \mp \cos l\,.\sec D\,.\,\rho\,.\sin P);$$

the upper sign before meridian passage, and in which l is the latitude of the observer, ρ the radius of the earth at his place, D the moon's declination, P her equatorial horizontal parallax, and m her daily motion in right ascension, in hours.

Making similar notation for a star,

$$s_* = h_* + e_* + i_{☽};$$

subtracting this from the preceding, and writing φ for $e_{☽} - e_*$, the clock acceleration in the interval, in time, between the moon and star,

$$s_{☽} - s_* = h_{☽} - h_* + \varphi \pm \frac{i_{☽}}{1 - 0{,}04166.m}(0{,}04166m \mp \cos l.\sec D.\rho.\sin P)$$

On a second meridian, to the west, a similar equation is found, with the variables accented. Taking the difference, and making

$$k_2 = \pm\left(\frac{i'_{☽} - i_{☽}}{1 - 0{,}04166}.0{,}04166.m \mp \frac{i'_{☽}.\sec D' - i_{☽}.\sec D}{1 - 0{,}04166\,.\,m}.\cos l.\rho.\sin P\right)$$

there will result

$$\Delta = s'_{☽} - s_{☽} = (h'_{☽} - h'_* + \varphi') - (h_{☽} - h_* + \varphi) \pm k_2 \quad . \quad . \quad (252)$$

Or, if there be but one observer with Ephemeris, then will $i_{☽} = 0$, and, omitting accents, the value of k_2 becomes,

$$k_1 = \pm \frac{i_{☽}}{1 - 0{,}04166\,m}\cdot(0{,}04166\,.\,m \mp \sec D\,.\cos l\,.\,\rho\,.\sin P).$$

$i_{☽}$, is found by the method of 13, Appendix II., p. 261.

in which Δ denotes the same as α, in Eq. (246), when a single observer, on an unknown meridian, employs the ephemeris elements, as given foı the next preceding passage over the first meridian, with those of his observations, to get the increase of right ascension requisite to find his approximate longitude l; and the same as $\alpha - \alpha_1$, in Eq. (247), when he employs either the interpolated or observed increase of right ascension for the meridian of which the approximate longitude is l, to correct its place. In the first case c_2 is the difference of the point's right ascension, as given for the next preceding upper and next following lower culmination over the first meridian ; in the second case v will be given by Eq. (251); and in both, H will be 12 hours of longitude.

Example.

OBSERVATIONS.—West Point, 1845, Feb. 18.

ζ Geminorum . . $6^h\ 54^m\ 41^s,75$			
δ " . . 7 10 38,97			
☽ W. Limb	$7^h\ 38^m\ 06^s,76$		
ζ Cancri 8 03 06,11			
3)22 08 26,83	7 22 48,94	. . .	$0^h\ 15^m\ 17^s,82$
$k_1 = 0$;	Clock rate, $+ 3^s$		$-0,03$

Nautical Almanac.—Greenwich, same date.

ζ Geminorum . . $6^h\ 54^m\ 57^s,41$			
δ " . . 7 10 54,36			
☽ W. Limb	$7^h\ 27^m\ 47^s,66$		
ζ Cancri 8 03 21,44			
3)22 09 13,21	7 23 04,40	. . .	0 04 43,26
	$a = \Delta =$	. . .	$0^h\ 10^m\ 34^s,53$

Then, Eq. (246),

	$H = 12^h\ 00^m\ 00^s,$	Log . .	4,6354837
	$a = 00\ 10\ 34,53$	" . .	2,8024520
Nautical Almanac	$c_2 = 00\ 25\ 41,18$	" a. c.	6,8121918
	$l = 4\ 56\ 28,$	" . .	4,2501275

Next, interpolate change of right ascension for l;

$$t = \frac{t_s - t'}{t_, - t'} = \frac{4^h\ 56^m\ 28^s}{12};$$

$4^h\ 56^m\ 28^s,$	Log . .	4,2501275
12 00 00,	" a. c.	5,3645163
t	" . .	9,6146436

Nautical Almanac.

Feb. 17,	L. C.	$7^h\ 01^m\ 56^s,27$			
			$25^m\ 51^s,39$		
" 18,	U. C.	7 27 47,66			$-10^s,21$
			$\Delta_1 = 25\ 41,18$	$\Delta_2 = \frac{1}{2}\Sigma$	$\Delta_3 = -0^s,25$
" "	L. C.	7 53 28,84			$-10,46$
			25 30,72		
" 19,	U. C.	8 18 59,56			

$$A = 25^m\ 41^s,18 + 05^s,17 - 0^s,02 = 25^m\ 46^s,33$$

$$B = -\ 05,17 + 00,06 \qquad = -\ 05,11$$

$$C = \ .\ .\ .\ .\ .\ .\ . \qquad -\ 00,04$$

Then, Eq. (249),

A	Log	3, 1893022		
t	"	9, 6146438		
		2, 8039460	Nos	$636^s, 72$
B	Log	$\bar{0}$, 7084209		
t^2	"	9, 2292876		
		$\bar{9}$, 9377085	Nos	− 0, 87
C	Log	$\bar{8}$, 6190933		
t^3	"	8, 8439314		
		$\bar{7}$, 4630247	Nos	− 0, 003
		a_1		635, 85
		$10^m\, 34^s, 53 = a$		634, 53
		$a - a_1 =$		$- 1^s, 32$

Again, Eq. (251),

A			Nos	$25^m\, 46^s, 33$
B	Log	$\bar{0}$, 7084209		
t	"	9, 6146438		
2	"	0, 3010300		
		$\bar{0}$, 6240947	Nos	− 4, 21
C	Log	$\bar{8}$, 6190933		
t^2	"	9, 2292876		
3	"	0, 4771213		
		$\bar{8}$, 3255022	Nos	− 0, 02
		$v =$		$25^m\, 42^s, 10$

Then, last term of Eq. (247),

H	Log	4, 6354837		
$a - a_1$	"	$\bar{0}$, 1205739		
v	" *a. c.*	6, 8118875		
$= - 36^s, 97$	"	$\bar{1}$, 5679451	Nos	00 36, 97

Eq. (247),

$$L = 4^h\, 56^m\, 28^s - 36^s, 97 = 4^h\, 55^m\, 51^s, 03$$

§ 729. It frequently happens that the moon cannot be observed on the middle wire, in which case she is far enough from the meridian to have a sensible parallax in right ascension; and as it may be very desirable not to lose the observation, this parallax must be computed and applied to the apparent hour angle from the middle wire, which is supposed to be nearly coincident with the meridian.

Denoting the hour angle by h, the parallax in hour angle by Δh, the geocentric latitude by l, the moon's declination by D, and her horizontal parallax by P, then, Appendix XI, p. 379,

$$\Delta h = \rho \cdot \cos l \cdot \sin P \cdot \sin h \cdot \sec D;$$

and to make this applicable to the case before us, h will denote the equatorial interval, in sidereal time, from the lateral to the central wire. This angle being small, its arc, expressed in seconds of time, may be taken for its sine, in which case, Δh will be in time-seconds, and the true distance of the moon's limb from the central wire, denoted by $h_{,}$, will be

$$h_{,} = h \cdot (1 - \rho \cdot \cos l \cdot \sin P \cdot \sec D);$$

and the reduction to the meridian, denoted by r, in time-seconds,

$$r = \pm \frac{h}{\cos D} \cdot \frac{1 - \rho \cdot \cos l \cdot \sin P \cdot \sec D}{1 - 0{,}04166 \cdot m},$$

in which m is the moon's daily motion in right ascension in hours. The upper sign, when the observation is before the middle wire. The quantities ρ and l are found from tables on pp. 336, 337.

§ 730. It also often happens that two observers do not use the same number of wires, or if they do, that the same stars are not observed at the same number. Such observations are not of equal weight. To find the relative value with which such observations should enter into the final determination, Professor Gauss has given the following formula, deduced from the principle of least squares.

Let the number of wires on which the moon is observed at one place be denoted by n, and at the other by n'; and let the number of wires at which the stars are observed at the first place be a, b, c, &c., and at the other be a', b', c', &c. Make

$$\frac{n\,n'}{n + n'} = \lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (253)$$

$$\frac{a\,a'}{a + a'} = \alpha, \quad \frac{b\,b'}{b + b'} = \beta, \quad \frac{c\,c'}{c + c'} = \gamma, \text{ \&c.} \quad . \quad . \quad . \quad (254)$$

$$\sigma = \alpha + \beta + \gamma + \text{\&c.} \quad . \qquad (255)$$

Then, if W denote the weight of each day's comparison, will

$$W = \frac{\sigma\lambda}{(\sigma + \lambda) z^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (256)$$

in which z is the same as $\frac{\alpha - \alpha_1}{v}$ in Eq. (247); and for the weight of the result of all the comparisons, we have

$$\Sigma W = \Sigma \frac{\sigma\lambda}{(\sigma + \lambda) z^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (257)$$

in which Σ expresses the sum.

Let e denote the probable error of observation, and E the probable error of the final result, then will

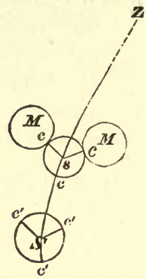
$$E = \frac{e}{\sqrt{\Sigma \frac{\sigma\lambda}{(\sigma + \lambda) z^2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (258)$$

§ 731. *Longitude by Telegraph.*—One of the simplest and most accurate methods for finding differences of longitude, is to telegraph to a western, the instant of a fixed star's culmination at an eastern station, and, conversely, to telegraph to the eastern the instant of culmination of the same star at the western station. The local times of both events being noted, the difference, as recorded at the same station, corrected for rate of time-keeper, gives the difference of longitude.

The instant of culmination of the moon's bright limb being also signalized in the same way, the difference of time, as recorded at the same station, corrected for rate, as before, gives the difference of longitude augmented by the limb's change in right ascension during the interval, and the excess of this interval over that for the fixed stars is the change itself. Thus the telegraph, where it connects stations remote from one another, gives the means for finding differences of longitude and for correcting the lunar ephemeris, and, therefore, the elements employed in the method of lunar culminations, for use at stations having no telegraphic connections.

§ 732. *Longitude by Solar Eclipse, or by Occultation.*—The following elegant and accurate solution of this most important problem is, in substance, due to Mr. Woolhouse; it first appeared in the Nautical Almanac for 1837.

Fig. 119.



Let M and S, be the moon and sun, in such geocentric positions as to appear in external tangential contact to an observer on the earth's surface; the local time of this observer will be that of beginning or ending of the local eclipse. Conceive a fictitious sun, s, at the distance of the moon, within and tangent to the visual cone that projects the true sun on the celestial sphere for this observer. This fictitious sun will be in contact with the moon; and any parallactic effect on the one, due to a change in the observer's place, will be equal to that on the other. Transport the observer to the centre of the earth; the moon and fictitious sun will appear to shift their places with respect to the true sun; but, being in actual, will remain in apparent contact. The apparent disk of the fictitious sun and of the moon will diminish; and the size and place of the latter will become those of the ephemeris at the instant of observation. The change of the fictitious sun's place, in reference to that of the true sun, will be the effect of relative parallax. Apply this parallax to the place of the true sun, and diminish his disk by a quantity equal to the diminution of the fictitious sun; the result will be the place and size of the latter body in apparent contact with the moon, to the observer at the central station. The ephemeris time of this contact, diminished by the local time of observation, will give the longitude of the observer.

Thus, the determination of terrestrial longitude, by a solar eclipse, is reduced to finding the ephemeris time when the true disk of the moon comes in contact with a disk of a given size, placed at a given place. The principle is the same for an occultation of a star by the moon.

In the case of a solar eclipse, the apparent time of observation, converted into arc, gives the hour angle of the sun's centre at that instant; and, as the declination of the sun is never subject to a very rapid daily variation, this element may be taken from the ephemeris, with sufficient accuracy, for the approximate local time on the meridian for which the ephemeris is constructed, deduced from an estimated longitude, or rough longitude, by account.

Take

$$\left.\begin{array}{l} \alpha = \text{right ascension,} \\ h = \text{hour angle,} \\ \delta = \text{declination,} \\ \sigma = \text{apparent semi-diameter,} \end{array}\right\} \text{of true sun;}$$

Take also

$$\left.\begin{array}{l} \alpha_0 = \text{right ascension,} \\ \delta_0 = \text{declination,} \\ \sigma_0 = \text{apparent semi-diameter,} \end{array}\right\} \text{of fictitious sun,}$$

$\Delta\alpha = \Delta h =$ relative parallax of moon in right ascension,

$\Delta\delta =$ " " " declination,

$\Delta\sigma =$ diminution of fictitious sun's semi-diameter;

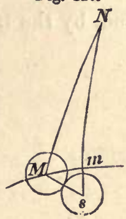
Then will

$$\alpha_0 = \alpha + \Delta h,$$

$$\delta_0 = \delta + \Delta\delta,$$

$$\sigma_0 = \sigma - \Delta\sigma.$$

Fig. 120.



Let N, be the north pole; M, the place of the moon at the instant of contact; m, her place when in conjunction with the fictitious sun, s.

Make

$(t) =$ any convenient ephemeris time, near this conjunction,

$(A) =$ moon's right ascension at (t),

$(D) =$ " declination at (t),

$A_1 =$ " relative motion in right ascension at (t),

$D_1 =$ " " " declination at (t),

$t_0 =$ time of true conjunction with fictitious sun, s,

$D_0 =$ declination of the point m at this time.

Then, employing the parenthesis to indicate the values of the several quantities at the time (t), we have

$$(\alpha_0) = (\alpha) + \Delta h, \qquad t_0 = (t) + \frac{(\alpha_0) - (A)}{A_1}.$$

$$(\delta_0) = (\delta) + \Delta\delta, \qquad D_0 = (D) + \frac{(\alpha_0) - (A)}{A_1} D_1.$$

Now, make

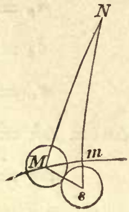
$$k = m\,s = D_0 - (\delta_0),$$

$$\Delta = M s,$$

$$90° + \eta = M m s,$$

$$90° + \psi = s M m;$$

Fig. 120. bis.



then will

$$\tan \eta = -\frac{D_1}{A_1 \cos (D)};$$

and, by the triangle $m\,M s$, considered as plane,

$$\cos \psi = \frac{k \cdot \cos \eta}{\Delta};$$

and, from the spherical triangle $N M s$,

$$\sin M N s = -\sin \Delta \frac{\sin (\eta + \psi)}{\cos (D)};$$

or as the small arcs are proportional to their sines,

$$M N s = -\Delta \frac{\sin (\eta + \psi)}{\cos (D)}.$$

And the time required for the moon to change her hour angle by this quantity, will be

$$\frac{M N s}{A_1} = -\frac{\Delta}{A_1} \cdot \frac{\sin (\eta + \psi)}{\cos (D)},$$

which, subtracted from t_c, will give the ephemeris time of observation. Denote this time by t, and we have

$$t = (t) + \frac{(\alpha_0) - (A)}{A_1} + \frac{\Delta}{A_1} \cdot \frac{\sin(\eta + \psi)}{\cos(D)}. \quad (259)$$

The longitude, from the meridian of the ephemeris, is found by the difference between this time and that of observation, previously making both apparent, or both mean time, by applying the equation of time; and it will be west or east, according as the ephemeris time is greater or less than that of observation.

To find $\Delta\alpha$, and $\Delta\delta$, take Eqs. (2), Appendix XI., p. 379, and write therein $\Delta\alpha$ for Δh, P for $\sin P$, $\Delta\delta$ for ΔD, δ for D and D', unity for $\cos \frac{1}{2}\Delta h$, and substitute for h its value $h' - \Delta h$; we find

$$\left.\begin{aligned} \Delta\alpha &= \rho \cdot P \cdot \frac{\cos l}{\cos \delta} \cdot \sin h, \\ \Delta\delta &= \rho \cdot P \cdot (\sin l \cdot \cos \delta - \cos l \cdot \sin \delta \cos (h - \tfrac{1}{2}\Delta\alpha); \end{aligned}\right\} \quad (260)$$

in which l denotes the central latitude; and, employing the method of solution in Appendix XI, page 381, we have

$$\left.\begin{aligned} &\Delta\alpha = \rho \cdot P \cdot \frac{\cos l}{\cos \delta} \cdot \sin h, \\ &(h) = h - \tfrac{1}{2}\Delta\alpha, \\ &\tan \theta = \cos (h) \cdot \cot l, \qquad \tan M = \frac{\sin \theta}{\cos (\theta + \delta)} \tan h, \\ &\tan \varepsilon = \tan (\theta + \delta) \cdot \cos M, \\ &\Delta\delta = \rho \cdot P \cdot \cos M \cdot \cos \varepsilon. \end{aligned}\right\} \quad (261)$$

To find $\Delta\sigma$, resume Eq. (27), substituting therein σ for s, σ' for s', $\cos (90° - \varepsilon)$ for $\cos Z$, unity for $\cos z$; and we have

$$\frac{\sigma'}{s} = \frac{\omega}{\omega - \rho \cdot P \cdot \sin \varepsilon};$$

subtracting unity from both members, clearing the fraction, writing $P - \pi$ for P, and then P' for $\rho\,(P - \pi)$, we have

$$\sigma' - \sigma = \Delta\sigma = \frac{\sigma \cdot P' \cdot \sin \varepsilon}{\omega - P' \cdot \sin \varepsilon} = \frac{\sigma}{10} \cdot \frac{P'}{10} \cdot \frac{100 \cdot \sin \varepsilon}{\omega - P' \cdot \sin \varepsilon}.$$

Taking the average value of P' in the denominator, say $57'\ 03''$, 5, and $\rho = 1$; and, expressing σ and P' in minutes, in which case $\omega = 3437, 45$, we may write

$$\Delta\sigma = \frac{\sigma}{10} \cdot \frac{P'}{10} \cdot f;$$

in which $\Delta\sigma$ will be expressed in seconds, if

$$f = \frac{100 \times 60 \cdot \sin \varepsilon}{3437', 45 - 57', 06 \cdot \sin \varepsilon}.$$

For an occultation of a star by the moon, the calculation will, in some respects, be slightly abridged. The characters A_1 and D_1 become the absolute motions of the moon in right ascension and declination; the semi-diameter σ, and its diminution $\Delta\sigma$, will reduce to zero; and the angle ε, which is only used to get $\Delta\sigma$, may be dispensed with; in which case it may be better to employ Eqs. (260) than Eq. (261); or Eqs. (261) may be modified into the following convenient expressions, by eliminating M and ε; viz.:

$$\left.\begin{array}{ll} \Delta\alpha = \rho \cdot P \cdot \dfrac{\cos l}{\cos \delta_0} \cdot \sin h, & (h) = h - \tfrac{1}{2}\Delta\alpha, \\[2ex] \tan \theta = \cos (h) \cdot \cot l, & \Delta\delta = \rho \cdot P \cdot \sin l \cdot \dfrac{\cos (\theta + \delta)}{\cos \theta}. \end{array}\right\} \quad (262)$$

It will be useful here to recapitulate the equations in a form suited to the facilitating of arithmetical calculation, and separately to arrange them for an eclipse of the sun, and an occultation of a star by the moon, to preserve distinctness.

I.—*Eclipse of the Sun.*

1. With the longitude by account find the corresponding Greenwich time, and thence from the ephemeris take out the sun's right ascension α, declination δ, and semi-diameter σ; the horizontal parallaxes P, π; also take out the moon's declination D roughly to the minute.

Reduce the latitude by the table on p. 336, and with ρ from the table on p. 337, Ap. XI, find

$$P' = \rho\,(P - \pi);$$

$h =$ apparent time of observation reduced into *arc*.

2. $$p = P' \cos l \sin h; \quad \varDelta h \text{ in min.} = [7.92082]\,\frac{p}{\cos D}; \quad (h) = h - \varDelta h;$$

$$\tan \theta = \cos (h) \cot l; \qquad G = \cos (h) \cos l;$$

$$\tan M = \frac{\sin \theta}{\cos (\theta + \delta)} \tan (h); \qquad \tan \varepsilon = \tan (\theta + \delta) \cos M;$$

$$B = \cos M \cos \varepsilon;$$

$$\text{check} \;\; . \;\; . \;\; \frac{\sin \theta}{\cos (\theta + \delta)} = \frac{G}{B};$$

$$\varDelta \delta = B \,.\, P'; \qquad \delta_0 = \delta + \varDelta \delta;$$

$$\varDelta \alpha \text{ in time} = [8.82391]\,\frac{p}{\cos \delta_0}; \qquad \alpha_0 = \alpha + \varDelta \alpha;$$

M to be in the same semicircle with h.

3. With ε find the corresponding factor f in the annexed table; then, using P and σ each in minutes,

$$\Delta \sigma \text{ in sec.} = \left(\frac{P'}{10}\right)\left(\frac{\sigma}{10}\right) . f;$$

and thence

$$\sigma_0 = \sigma - \Delta \sigma$$
$$s = [9.43537]\,P.$$

$$\text{For} \left\{ \begin{array}{l} \text{partial} \\ \text{total or annular} \end{array} \right\} \text{phase, } \Delta = \left\{ \begin{array}{l} s + \sigma_0 \\ s \sim \sigma_0 . \end{array} \right.$$

ε	Factor f for diminution of ☉'s semi-diam.	
°		
0	0·01	
		+·30
10	0·31	
		·30
20	0·61	
		·28
30	0·89	
		·26
40	1·15	
		·22
50	1·37	
		·17
60	1·54	
		·13
70	1·67	
		·08
80	1·75	
		+·02
90	1·77	

4. In the hourly ephemeris of the moon, fix on a convenient time (t) at which the moon's right ascension is near to α_0, and for this time take out the right ascension (A) in time, the declination ($\varDelta'$), and their hourly va-

riations; also the sun's right ascension (α), declination (δ), and their hourly variations. Then,

$$A_1 = \text{hourly var. } (A) - \text{hourly var. } (\alpha) \text{ in time;}$$
$$D_1 = \text{hourly var. } (D) - \text{hourly var. } (\delta) \text{ in arc;}$$
$$(\alpha_0) = (\alpha) + \Delta\,\alpha;$$
$$(\delta_0) = (\delta) + \Delta\,\delta.$$

5.
$$m = \frac{(\alpha_0) - (A)}{A_1}; \qquad t_0 = (t) + m\,[3.55630];$$
$$D_0 = (D) + m\,.\,D_1; \qquad k = D_0 - (\delta_0);$$
$$n = [1.17609]\,A_1 \cos (D);$$
$$\tan \eta = -\frac{D_1}{n}; \qquad \cos \psi = \frac{k \cos \eta}{\Delta}.$$

$$\text{Corresponding Greenwich mean time} = t_0 + [3.55630]\frac{\Delta}{n} \sin (\eta \mp \psi),$$

n to have a different sign from D_1:

upper / under sign when an immersion / emersion is observed.

II.—*Occultation of a Star by the Moon.*

6. With the estimated longitude find the corresponding Greenwich time, and thence take out the moon's horizontal parallax P, and her declination D, roughly to the minute; also,

sid. time = apparent time + ☉'s right ascension; or,
sid. time = mean time + sid. time mean noon, from p. III. of ephemeris + accel. on Greenwich mean time;

$$h = \text{sid. time} - \alpha, \text{ in arc};$$
$$P' = \rho\,P;$$

α being the star's right ascension.

7.
$$p = P' \cos l \sin h; \quad \Delta\,h \text{ in min.} = [7.92082]\frac{p}{\cos D}; \quad (h) = h - \Delta\,h;$$
$$\varkappa = P' \sin l \cos \delta; \quad \varkappa' = P' \cos l \sin \delta \cos (h); \quad \delta_0 = \delta + \varkappa - \varkappa';$$
$$\Delta\,\alpha \text{ in time} = [8.82391]\frac{p}{\cos \delta_0}; \qquad \alpha_0 = \alpha + \Delta\,\alpha.$$

8. In the hourly ephemeris of the moon fix on a convenient time (t) at which the moon's right ascension is near to α_0, and for this time take out

the right ascension (A), the declination (D), and their hourly variations A_1, D_1. Then,

$$m = \frac{\alpha_0 - (A)}{A_1}; \qquad t_0 = (t) + [3.55630]\, m;$$

$$D_0 = (D) + m \,.\, D_1; \qquad k = D_0 - \delta_0;$$

$$n = [1.17609]\, A_1 \cos (D);$$

$$\tan \eta = -\frac{D_1}{n}; \qquad \cos \psi = [0.56463] \frac{k \cos \eta}{P}.$$

$$\text{Corresponding Greenwich mean time} = t_0 + [2.99167] \frac{P}{n} \sin (\eta \mp \psi).$$

Practical Rules for Calculating the Longitude from an Observed Occultation.

With the estimated longitude find the corresponding Greenwich time roughly to the minute, and for this time take out from the ephemeris the moon's declination roughly to the minute, her horizontal parallax to the tenth of a second, and the sun's right ascension in time to the nearest second. To the sun's right ascension add the apparent time of the observation, which will give the right ascension of the meridian. The difference between this right ascension and that of the star will give the hour angle of the star in time, which must be reduced into arc in the usual manner: it will be

$$\left.\begin{matrix}\text{W.}\\ \text{E.}\end{matrix}\right\} \text{ when R. A. of meridian is } \left\{\begin{matrix}\text{greater}\\ \text{less}\end{matrix}\right\} \text{ than R. A. of } *.$$

Reduce the latitude of the place by subtracting the correction found in the table in Appendix XI., p. 336, for which the nearest correction found in the table will be sufficient.

To the proportional logarithm of the moon's horizontal parallax, add the correction answering to the latitude in the following series:

Lat. .	0°	11°	19°	24°	29°	34°	38°	42°	46°	50°	54°	59°	64°	69°	77°	90°
Corr. .	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	

To the proportional logarithm of the horizontal parallax, so corrected, add the log. secant of the reduced latitude and the log. cosecant of the hour angle. To the sum (S_1) add the log. cosine of the moon's declination and the constant log. 0.3010. The result will be the prop. log. of an arc, which, subtracted from the hour angle, will give the hour angle corrected.

To the corrected prop. log. of the horizontal parallax, add the log. secant

of the *'s declination, and the log. cosecant of the reduced latitude. To the same log. add the log. cosecant of the *'s declination, the log. secant of the reduced latitude, and the log. secant of the hour angle corrected. These sums will be the prop. logs. of two arcs.

The former arc to have the same name as the latitude.

The latter to have

$$\left.\begin{array}{l}\text{a different name from}\\ \text{the same name as}\end{array}\right\}\text{ the dec. when the h. angle is }\left\{\begin{array}{l}\text{less}\\ \text{greater}\end{array}\right\}\text{ than }90^\circ$$

The sum of these two arcs, having regard to their names, will give the correction to be applied to the *'s declination to get the declination corrected.

To the sum (S_1) add the constant log. 1.1761, and the log. cosine of the *'s declination corrected; the sum will be the prop. log. of an arc in time, to be

$$\left.\begin{array}{l}\text{added to}\\ \text{subtracted from}\end{array}\right\}\text{ the }*\text{'s R. A., when it is }\left\{\begin{array}{l}\text{west}\\ \text{east}\end{array}\right\}\text{ of the meridian,}$$

to get the *'s right ascension corrected.

In the hourly ephemeris of the moon, fix on a convenient time at which her right ascension is near to that of the star corrected; and, for this time, take out the right ascension, the declination, and their hourly variations.

Subtract the common log. of the difference between the corrected right ascension of the star and the right ascension of the moon, from the common log. of the hourly motion in right ascension; to the remainder add the constant log. 0.4771; to the same remainder add the prop. log. of the hourly motion in declination. The former sum will be the prop. log. of a time to be

$$\left.\begin{array}{l}\text{added to}\\ \text{subtracted from}\end{array}\right\}\text{ the assumed time when }*\text{'s R. A. is }\left\{\begin{array}{l}\text{greater}\\ \text{less}\end{array}\right\}\text{ than ☽'s R. A.}$$

to get the time corrected.

The latter will be the prop. log. of a correction of the ☽'s declination, to be applied with

$$\left.\begin{array}{l}\text{the same name as}\\ \text{a different name from}\end{array}\right\}\text{ hourly var. when }*\text{'s R. A. is }\left\{\begin{array}{l}\text{greater}\\ \text{less}\end{array}\right\}\text{ than ☽'s R. A.}$$

To the common log. of the hourly motion in right ascension, add the log. cosine of the ☽'s corrected declination; and to the sum (S_2) add the prop. log. of the hourly motion in declination and the constant log. 7.1427.

The result will be the log. cotangent of the first orbital inclination,* and must take

$$\left.\begin{array}{l}\text{the same name as}\\ \text{a different name from}\end{array}\right\}\text{hourly motion in dec. when} * \text{is}\left\{\begin{array}{l}\text{north}\\ \text{south}\end{array}\right\}\text{of}\ \text{☽}$$

To the prop. log. of the difference between the star's declination corrected and the moon's declination corrected, add the constant log. 9.4354, and the log. secant of the preceding orbital inclination; and from the sum deduct the prop. log. of the horizontal parallax. The remainder will be the log. secant of the second orbital inclination,† which must have the name

$$\left.\begin{array}{l}\text{S.}\\ \text{N.}\end{array}\right\}\text{when the observation is an}\left\{\begin{array}{l}\text{immersion}\\ \text{emersion.}\end{array}\right.$$

Add together the two orbital inclinations, having proper regard to their names; and to the log. cosecant of this sum add the preceding sum (S_2), the prop. log. of the horizontal parallax, and the constant log. 8.1844. The sum will be the prop. log. of a correction to be applied to the time corrected to get the mean time at Greenwich: it must be

$$\left.\begin{array}{l}\text{added}\\ \text{subtracted}\end{array}\right\}\text{when the sum of the orbital inclinations is}\left\{\begin{array}{l}\text{N.}\\ \text{S.}\end{array}\right.$$

By applying the equation of time from p. II. of the ephemeris, there will result the Greenwich apparent time, and the difference between it and the apparent time of observation will show the longitude of the place from Greenwich; it will be

$$\left.\begin{array}{l}\text{W.}\\ \text{E.}\end{array}\right\}\text{when the Greenwich time is}\left\{\begin{array}{l}\text{greater}\\ \text{less}\end{array}\right\}\text{than the observed.}$$

Examples.

I. SOLAR ECLIPSE.

For a solar eclipse, take the example directly calculated in Appendix XI., page 412:

Suppose the beginning of the solar eclipse on May 15, 1836, to be observed to take place at $1^h\ 36^m\ 35^s{\cdot}6$ P. M., apparent time, in latitude 55° 57′ 20″ N., and longitude about 12^m W.

---

* With the parallel of declination. † With the moon's limb.

Here we have

	h.	m.
Observed apparent time .	1	36·6
Longitude		12·0
Greenwich apparent time .	1	48·6
Equation of time . . .		3·9
Greenwich mean time . .	1	44·7

$h = +1^h\ 56^m\ 35^s{\cdot}6$

$= +\ 24°\ 8'{\cdot}9$

We hence take from the ephemeris, $\alpha = 3^h\ 29^m\ 19^s$, $\delta = +18°\ 57'{\cdot}t$ $\sigma = 15'\ 49''{\cdot}9$, $D = +19°\ 19'$, $P = 54'\ 24''{\cdot}4$, $\pi = 8''{\cdot}5$, $P - \pi = 54'\ 15''{\cdot}9$.

Latitude	+ 55° 57′ 20″	
Reduction	10 28	
l .	+ 55 46 52	$\rho = 9{\cdot}99902$

	$P-\pi$ 3·51267		cos (h)	+9·96060	.	+9·96060
	ρ . 9·99902		cot l	+9·83256	cos l	+9·75001
	P' . 3·51169	θ +31° 50′·7	tan θ	+9·79316	G	+9·71061
	cos l 9·75001	δ +18 57·6	sin θ	+9·72231		
	sin h +9·61183	$\theta+\delta$ +50 48·3	cos	+9·80069	B	+9·78899
	p +2·87353 (1)			+9·92162	check	+9·9216?
	cos D 9·97484		tan (h)	+9·64936		
	+2·89869		tan M	+9·57098		
h +24° 8′·9	const. 7·92082		cos M	+9·97180	.	+9·97180
Δh + 6·6	+0·81951		. tan $(\theta+\delta)$	+0·08861	cos ϵ	+9·81719
(h) +24 2·3		ϵ +48° 58′·3	tan ϵ	+0·06041	B	+9·78899
					P'	+3·51169
δ +18 57·6						
+ 33·3			$\Delta\delta$	+ 33′ 18″·4	.	+3·30068
δ_0 +19 30·9	cos 9·97430 (2)					
			σ .	15′ 49″·9		
	+2·89923 (1)—(2)		$\Delta\sigma$	11 ·6	P .	3·51380
	const. 8·82391				const.	9·43537
			σ_0 .	15 38 ·3		
	log. . +1·72314		s .	14 49 ·6		2·94917
	$\Delta\alpha$ + 0h 0m 52s·86		Δ .	30 27 ·9		
	α 3 29 19					
	α_0 3 30 12					

By inspecting the hourly ephemeris of the moon's right ascension on May 15th with $\alpha_0 = 3^h\ 30^m\ 12^s$, the most eligible time to assume is evidently $(t) = 3^h\ 0^m\ 0^s$; at this time we have $(A) = 3^h\ 30^m\ 42^s{\cdot}84$, $(A_1) = 2^m\ 0^s{\cdot}68$, $(D) = +19°\ 31'\ 34''{\cdot}0$, $(D_1) = +9'\ 55''{\cdot}2$, $(\alpha) = 3^h\ 29^m\ 31^s{\cdot}57$, $(\alpha_1) = +9^s{\cdot}89$, $(\delta) = +18°\ 58'\ 21''{\cdot}4$, $(\delta_1) = +34''{\cdot}8$: with these we proceed as follows:

	m.	s.		′	″
(A_1) . . .	2	0·68	(D_1) . .	+ 9	55·2
(α_1) . .		9·89	(δ_1) . .	+	34·8
A_1 . . .	1	50·79	D_1 . .	+ 9	20·4

	h. m. s.		° ′ ″
(α) .	3 29 31·57	(δ) +	18 58 21·4
$\triangle\alpha$ +	0 52·86	$\triangle\delta$ +	33 18·4
(α_0) .	3 30 24·43	(δ_0) +	19 31 39·8
(A) .	3 30 42·84		
$(\alpha_0)-(A)$.	− 0 18·41		
log . . .	− 1·26505		
A_1 .	2·04450	D_1 . .	+ 2·74850 (1)
m . .	− 9·22055		− 9·22055
const.	3·55630	log. .	− 1·96905
log. .	− 2.77685		− 0° 1′ 33″·1
	− 0h 9m 58s·2	(D)	+ 19 31 34 ·0
(t)	3 0 0	D_0	+ 19 30 0 ·9
t_0 +	2 50 1 ·8	(δ_0)	+ 19 31 39 ·8
		k	− 1 38 ·9
		cos (D) .	9·97428
		A_1 . .	2·04450
		const. . .	1·17609
		n . .	3·19487 (2)
η . . .	− 19° 41·2	tan η . . .	− 9·55363 (1) — (2)
		cos η . .	+ 9·97384
		k . .	− 1·99520
			− 1·96904
		$\triangle$ . .	3·26196
ψ . . .	+ 92 55·2	cos ψ . . .	− 8·70708
$\eta-\psi$ . . .	− 112 36·4 . .	sin . . .	− 9·96528
		$\triangle$ . .	3·26196
		const. . .	3·55630
			− 6·78354 (3)
corr.	− 1h 4m 38s·5		− 3·58867 (3) — (2)

t_0 + corr.	+ 1 45 23 ·3	Greenwich mean time.
	3 56 ·0	Equation of time.
	1 49 19 ·3	Greenwich apparent time.
	1 36 35 ·6	Observed " "
Longitude .	12 43 ·7	W. of Greenwich.

II. OCCULTATION OF A STAR.

Suppose, at Bedford, on January 7, 1836, in latitude 52° 8′ 28″ N., the immersion of *i* Leonis to be observed at $10^h\ 39^m\ 22^s.4$ P. M., apparent time, and the estimated longitude to be about $0^h\ 1^m$ W. Required the longitude?

	h. m.		° ′ ″
Apparent time (observation) .	10 39	Latitude . .	N. 52 8 28
Longitude	0 1 W.	Reduc. . . .	10 57
Apparent time (Greenwich .	10 40		N. 51 57 31
Equation of time	7	Reduced or geocentric latitude.	
Mean time (Greenwich) . .	10 47		

For Jan. 7, at $10^h\ 47^m$, we find, from the Ephemeris, ☉'s R. A. = $19^h\ 12^m\ 40^s$ ☽'s dec. = N. 15° 50′, and ☽'s equ. hor. par. = 56′ 1″·9.

	h. m. s.		
☉'s R. A.	19 12 40	P. L. ☽'s hor. par. .	0·5068
App. time	10 39 22	corr. for lat. . . .	9
R. A. meridian	5 52 2	P. L. corr^d. hor. par .	0·5077
" ✱	10 23 26	sec. red. lat. . . .	0·2103
✱'s hour angle E. { in time .	4 31 24	cosec. hour angle .	0·0333
✱'s hour angle E. { in arc .	67° 51′	sum (S_1)	0·7513
		cos. ☽'s dec. . . .	9·9832
		const. log. . . .	0·3010
corr^n. . .	17 . .	P. L. corr^n. . . .	1·0355
✱'s hour angle E. corr^d. . .	67 34		

P. L. corr^d. hor. par. .	0·5077		0·5077
sec. ✱'s dec. . . .	0·0150 .	cosec.	0·5876
cosec. red. lat. . . .	0·1037 .	sec.	0·2103
° ′ ″		sec. corr^d. hour angle	0·4184
N. 0 42 33·0	P. L. 0·6264		
S. 0 3 23·9	. . . . . .	P. L.	1·7240
corr^n. . . N. 0 39 9·1		sum (S_1)	0·7513
✱'s dec. . N. 14 58 38·8		const. log. . . .	1·1761
✱'s dec. corr^d. N. 15 37 47·9	. . . . . .	cos.	9·9836
corr^n. . . .	$0^h\ 2^m\ 12^s.56$	P. L. corr^n. . . .	1·9110
✱'s R. A. . .	10 23 26·39		
✱'s R. A. corr^d.	10 21 13·83		

On referring with the ✱'s corrected R. A. to the hourly ephemeris of the moon, it will evidently be most convenient to take out the data at 11^h; for this time we have ☽'s R. A. = $10^h\ 20^m\ 58^s.47$, hourly motion ☽'s R. A. = $2^m\ 2^s.9$, ☽'s dec. = N. 15° 47′ 11″·0, hourly motion ☽'s dec = S. 11′ 41″·5.

	h. m. s.
✱'s corrd. R. A.	10 21 13·83
☽'s R. A. . .	10 20 58·47
diff. . . .	0 15·36
common log . .	1·1864
com. log. h. m. ☽'s R. A.	2·0896

Remainder .	0.9032	. . . .	0·9032
const. log.	0.4771	P. L. h. m. ☽'s dec. . . .	1·1874
corrn. . . 0 7 29·9 (h. m. s.) P. L.	1·3803	corrn. . S. 0° 1′ 27″·7 P. L.	2·0906
Time assumed 11 0 0		☽'s dec. . N. 15 47 11·0	
Time corrd. . 11 7 29·9		☽'s dec. corrd. N. 15 45 43·3	

com. log. h. m. ☽'s R. A. .	2·0896	✱'s corrd. dec. . .	N. 15° 37′ 47·9
cos. ☽'s corrd. dec. . . .	9·9834	☽'s " " . .	N. 15 45 43·3
sum (S_2)	2·0730	diff. (✱ S. of ☽)	7 55·4
P. L. h. m. ☽'s dec. . . .	1·1874	P. L.	1·3563
const. log.	7·1427	const. log. . . .	9·4354
1st Orb. incl. N. 21° 34′ cot.	0·4031	. sec	0·0315
			0·8232
		P. L. ☽'s hor. par. .	0·5068
2nd Orb. incl. S. 61 9	. . . . . .	sec	0·3164
sum . . . S. 39 35	. . . . . .	cosec.	0·1957
		sum (S_2) . . .	2·0730
		P. L. ☽'s hor. par. .	0·5068
		const. log. . . .	8·1844
corrn.	h. m. s. 0 19 44·5	P. L.	0·9599
Time corrd.	11 7 29·9		
Greenwich mean time	10 47 45·4		
Equation of time . .	6 31·0		
Greenwich app. time	10 41 14·4		
Observed " "	10 39 22·4		
Longitude . . .	1 52·0 W		

P. S.—The principle of reversing the effect of the relative horizontal parallax on the position of the sun, instead of using the actual effect on the position of the moon, may be advantageously employed in the direct calculation of an eclipse for a particular place. It will only be necessary to use the parallaxes for the sun viewed as an apparent position, and to diminish the semi-diameter by the amount derived from the table on page 360. Thus, it appears, at the beginning of the eclipse, for instance, that the contact may be mathematically tested in two ways. First, we may apply the actual effects of the parallax to the true position of the moon, then augment her semi-diameter, and thus establish a contact of the limbs. But, if we reverse the operation, and consider the sun to be an apparent body under the influence of the relative parallax, then clearing it from this supposed

influence by reversing the parallax, and diminishing the semi-diameter, a contact will similarly be established with the true limb of the moon; and this principle, in its application to solar eclipses, possesses an advantage similar to that derived in the case of an occultation, by considering the star as an apparent place. (See Appendix XI., page 399)*

The formulæ, Nos. 2, 3, 4, and 5, pp. 406, 407, may, according to this method, be supplied by the following:

2. $P' = \rho(P - \pi)$; $m = P' \cos l$;

$Q_1 = [9.4180]$; $Q_2 = [9.4180]\, m \sin \delta$;

$$s = [9.43537]\, P.$$

3
$$k = \frac{m}{\cos D};$$

$$\Delta h \text{ in minutes} = [7.92082]\, k \sin h;$$

$$(h) = h - \Delta h;$$

$\tan \theta = \cos (h) \cot l$; $G = \cos (h) \cos l$,

$\tan M = \frac{\sin \theta}{\cos (\theta + \delta)} \tan (h)$; $\tan \varepsilon = \tan (\theta + \delta) \cos M$;

$$B = \cos M \cos \varepsilon;$$

$$\text{check} \quad . \quad . \quad \frac{\sin \theta}{\cos (\theta + \delta)} = \frac{G}{B};$$

$$\Delta \delta = B \,.\, P';$$

$$\sigma_0 = \sigma - \text{diminution for } \varepsilon;$$

$$\text{For} \left\{ \begin{array}{l} \text{partial} \\ \text{total or annular} \end{array} \right\} \text{phase}, \angle' = \left\{ \begin{array}{l} s + \sigma_0 \\ s \sim \sigma_0. \end{array} \right.$$

4. $k_0 = \frac{m}{\cos \delta_0}$; $\Delta \alpha = k_0 \sin h$;

$\Delta \alpha_1 = Q_1 k_0 \cos h$; $\Delta \delta_1 = Q_2 \sin (h)$.

5. $\delta_0 = \delta + \Delta \delta$; $\alpha' = \alpha - \Delta \alpha$;

$y = (\alpha - \Delta \alpha) \cos D$; $y_1 = (\alpha_1 - \Delta \alpha_1) \cos D$;

$x = (D + \alpha' \text{ corr.}) - \delta_0$; $x_1 = D_1 - \Delta \delta_1$.

---

* This was inadvertently ascribed to Carlini. Professor Henderson, by whom a paper has appeared upon this very point in the *Quarterly Journal* for 1828, page 411, informs me that the method has been long in practice, and that it was employed at an early period by Dr. Maskelyne.

§ 734. *Longitude by Eclipses of Jupiter's Satellites.*—The eclipses of Jupiter's satellites are computed in advance, and the times of occurrence inserted in the Nautical Almanac, to facilitate the determination of terrestrial longitude. After ascertaining, by inspection, about the time an eclipse begins and ends, the satellites are watched with a good telescope, and the precise local time of entrance into and departure from the shadow noted as nearly as possible. The time given in the Almanac, diminished by this observed local time, is the longitude; west, when the difference is positive, east when negative. This method for finding longitude is defective, for reasons stated in § 497.

CALENDAR.

§ 735. To divide and measure time and to note the occurrence of events in a way to give a distinct idea of their order of succession and the intervals of time between them, is the purpose of *Chronology.*

§ 736. All measurements require standard units. These units are, for the most part, purely arbitrary, and are equally convenient in practice. But such is not the case in chronology. Time is divided and marked by phenomena which are beyond our control, and which indeed regulate our wants and occupations. The alternation of day and night forces upon us the *solar day* as a natural unit of time.

§ 737. To avoid the use of numerous figures in the expression of great magnitudes, all measurements must have their scales of large and small units, and usually the selection of the larger is as arbitrary as the smaller; but here the phenomena of nature again interpose, and the periodical return of the seasons, upon which all the more important arrangements and business transactions of life depend, prescribes the tropical year as another and higher order of unit in chronology.

§ 738. But the solar day and tropical year are both variable, and are therefore wanting in all the essential qualities of standards. Neither are they commensurable the one with the other; they are on this account unfit units for the same scale. In the measurement of space, for instance, each unit is constant, and one is an aliquot part of another—a yard is equivalent to three feet, a foot to twelve inches, &c. But a year is no exact number of days, nor an integer number and any exact fraction, as a third or a fourth, even; but the surplus is an incommensurable fraction which possesses the same kind of inconvenience in the reckoning of time that would arise in that of money with gold coins of 101 dimes and odd cents, and a fraction over. For this there would be no remedy but to

keep an accurate register of the surplus fractions, and when they amount to a whole unit, to cast them over to the integer account. To do this in the simplest and most convenient manner in the reckoning of time, is the object of the *calendar*.

§ 739. A calendar is, therefore, a classification of the natural and other divisions of time, with such rules for their application to chronology as shall take into account every portion of duration without recording any one portion twice.

These divisions are years, months, weeks, days, and certain periods, to be noticed presently, and which are chiefly important in the use made of them in fixing upon a common epoch or origin of reference.

§ 740. *Julian Calendar.*—The years are denominated *as years current, not as years past*, from the midnight between the 31st of December and 1st of January, immediately subsequent to the birth of Christ, according to the chronological determination of that event, and this origin is designated by the letters A. D. or B. C., according as the year is subsequent or previous. Every year whose number is not divisible by four without a remainder, consists of 365 days, and every year which is *so* divisible of 366. The additional day in every fourth year is called the *Intercalary day*. The years which consist of 365 days are called *Common years;* those which consist of 366 days are called *Bissextile years*, and frequently *Leap* years. The mean length of the year by this rule is obviously $365\frac{1}{4}$ days, and the mode of reckoning time by this unit in the way just described is called the *Julian Calendar*.

§ 741. The year is divided into 12 months of unequal length. They are named, in order of succession, January, February, March, April, May, June, July, August, September, October, November, and December. January, March, May, July, August, October, and December, have each 31 days; each of the others except February has 30, and February has in a common year 28 and in a bissextile year 29; so that the intercalary day is added to February. The weeks consist of seven days, named in order, Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, and Saturday.

§ 742. *Gregorian Calendar.*—The Julian year consists of 365.25 days; the tropical year of 365.24224, making the Julian longer than the tropical by 0.00776 of a day, and causing the seasons to begin earlier and earlier every year as designated by the Julian dates. In process of time the seasons would therefore correspond to opposite dates of the year, and as this was likely to interfere with the times of holding certain church festivals, Pope Gregory XIII. determined upon a reformation of the Julian calendar.

§ 743. In A. D. 325, the seasons, festivals, and Julian dates corresponded with one another, according to church rule. The reformation was effected in 1582. Now $(1582 - 325) \times 0^d.00776 = 9^d.6243$. Again, $0^d.00776 \times 400 = 3^d.104$. The Pope ordered that the day following, the 4th of October, 1582, should be called the 15th instead of the 5th. This brought the date of the sun's entering the vernal equinox to what it was in 325, the time of holding the Council of Nice. And to secure this coincidence in future, he also ordered that *three* intercalary days should be omitted every four hundred years, the omissions to take place in those centennial years which are not divisible by 400; so that 1700, 1800, and 1900, which by the Julian mode of reckoning are bissextile, are made by the Gregorian common years. There is, therefore, at the present time, viz., in the 19th century, a difference of 12 days between the Julian and Gregorian dates. The mode of reckoning by the Julian calendar is called *Old*, and that by the Gregorian *New Style.* New style is followed throughout Christendom except in Russia, where the old style is preserved.

§ 744. *Solar Cycle.*—This is a period of 28 Julian years, after the lapse of which the same days of the week in the Julian system would return to the same days of each month throughout the year. For four such years consist of 1461 days, which is not a multiple of 7, but 7 times 4 or 28 years is a multiple of 7. The place in this cycle for any year of A. D. is found by adding 9 to the year, dividing by 28, and taking the remainder. When there is no remainder, the number sought is 28.

§ 745. *Lunar Cycle.*—This is a period of 19 years or 235 lunations which differ from 19 Julian years only by about an hour and a half; so that, supposing the new moon to happen on the first of January in the first year of the lunar cycle, it will happen on that day or within a very short time of its beginning or ending again after the lapse of 19 years. The number of the year of the lunar cycle is called the *golden number*, to find which add 1 to the number of the year A. D., and take the remainder after dividing by 19. If there be no remainder, the golden number will be 19. The golden number is used in ecclesiastical dates to determine the civil date of Easter.

§ 746. *Cycle of Indiction.*—This is a period of 15 years, used in the courts of law and in the fiscal organization of the Roman empire, and thence introduced into legal dates as the golden number into the ecclesiastical. To find the place of any year of A. D. in the cycle of indiction, add 3, divide by 15, and take the remainder. If there be no remainder, the number sought will be 15.

§ 747. *Julian Period.*—The product of 28, 19, and 15 is 7980. This is called the *Julian Period;* and it is obvious that after this period, the years of the solar, lunar, and indiction cycles will recur in the same order; that is, each year will hold the same place in all the three cycles as the corresponding year in the previous period.

§ 748. As no common factor exists in the numbers 28, 19, and 15, it is plain that no two years in the Julian period can agree in its three component cycles, and to specify the number of a year in each of the latter is to specify the number of the year in the Julian period, which now embraces the entire authentic chronology. The first year of the current Julian period, or that of which the number of the three subordinate periods is 1, was the year B. C. 4713, and noon of the 1st of January of that year, for the meridian of Alexandria in Egypt is the *chronological epoch* to which all historical eras are most readily referred, by computing the number of integer days intervening between it and Alexandria noon of the days which serve as the respective epochs of these eras. The meridian of Alexandria is chosen, because it is that to which Ptolemy refers the commencement of the era of Nabonassar, the basis of all his calculations.

§ 749. Given the year of the Julian period, those of the subordinate cycles are found as above. Conversely, given the year of the solar, lunar, and indiction cycles, to determine the year of the Julian period, proceed as follows, viz.: Multiply the number of the year in the solar cycle by 4845, in the lunar by 4200, and in the indiction by 6916, and divide the sum of the products by 7980, and the remainder will be the year of the Julian period sought.

§ 750. A date, whether of a day or year, always expresses, as before remarked, the day or year *current,* not *elapsed;* and the designation of a year by A. D. or B. C. is to be regarded as the *name* of that year, and not as a mere *number designating the place of the year in a scale of time.* Thus, in the date January 5, B. C. 1, January 5th does not mean that 5 days in January have elapsed, but that 4 have elapsed, and the 5th is current. And B. C. 1, indicates that the *first* day of the year so named (the first current before Christ) preceded the first day of the common era by one year. The scale A. D. and B. C. is not continuous; the year 0, is wanting in both parts, so that supposing the common reckoning correct, our Saviour was born in the year B. C. 1.

§ 751. *Epact.*—The mean age of the moon at the commencement of a year is called the *epact.* It is a name given to the interval of time between the first of the year and the next preceding mean new moon: it is expressed in days, hours, minutes, and seconds. Its use is to find the days

of mean new and full moon throughout the year, and thence the dates of certain church festivals.

§ 752. *Equinoctial Time.*—Astronomical time reckons from *noon* of the current day; civil, from the preceding *midnight.* Astronomical and civil dates coincide, therefore, only during the first half of the astronomical and last half of the civil day. Were this the only cause of discrepancy, it might be remedied by shifting the astronomical epoch to coincide with the civil. But there is an inconvenience to which both are liable, inherent in the nature of the day itself, which is a local phenomenon, and commences at different instants of absolute time under different meridians. In consequence, all astronomical observations require to be given, to render them comparable with one another, in addition to their date, the longitude of the place of observation from some known meridian. But even this does not meet the whole difficulty, for when it is Monday, 1st of January, of any year, in one part of the world, it will be Sunday, 31st December, of the preceding year, in another part of the world, so long as time is reckoned by local hours.

The equivoque can only be avoided by reckoning time from an epoch common to all the earth. Such an epoch is that which marks the passage of an imaginary sun having a mean motion equal to that of the true sun, through a mean vernal equinox receding uniformly upon the ecliptic with a motion equal to the mean motion of the true equinox. Time reckoned from this epoch is called *equinoctial time.* Equinoctial time is therefore the mean longitude of the sun converted into time at the rate of 360° to the tropical year.

APPENDIX.

APPENDIX.

APPENDIX I.

ELEMENTS OF THE PRINCIPAL PLANETS.

SIGNS AND NAMES OF PLANETS.	MEAN MOTION.	PERIODIC TIMES.	MEAN DISTANCES.	ECCENTRICITY.	LONGITUDE OF PERIHELION.	MEAN LONGITUDE OF EPOCH.	LONGITUDE OF ASCENDING NODE.	INCLINATION.	EPOCH, PARIS.
					° ′ ″	° ′ ″	° ′ ″	° ′ ″	
☿ Mercury	14732″.419	0.2408	0.3870985	0.2056063	74 20 42	110 13 18	45 57 38	7 0 5	1800 1st Jan.
♀ Venus	5767.668	0.6152	0.7233317	0.0068618	128 43 6	146 44 56	74 51 41	3 23 29	*id.*
⊕ Earth.......	3548.193	1.0000	1.000000	0.01679226	99 30 29	100 53 30	0 0 0	0 0 0	*id.*
♂ Mars	1886.518	1.8807	1.523691	0.0932168	332 22 51	233 5 34	47 59 38	1 51 6	*id.*
♃ Jupiter	298.989	11.8600	5.202767	0.0481621	11 7 38	81 54 49	98 25 45	1 18 52	*id.*
♄ Saturn......	120.435	29.4600	9.538850	0.0561505	89 8 20	123 6 29	111 56 7	2 29 36	*id.*
♅ Uranus	42.233	84.0100	19.1824	0.0466	167 30 24	173 30 37	72 59 21	0 46 28	*id.*
♆ Neptune	21.554	164.6200	30.04	0.0087195	47 14 37	335 8 58	130 6 52	1 46 59	*id.*

APPENDIX II.

ASTRONOMICAL INSTRUMENTS.

Astronomical Clock and Chronometer.

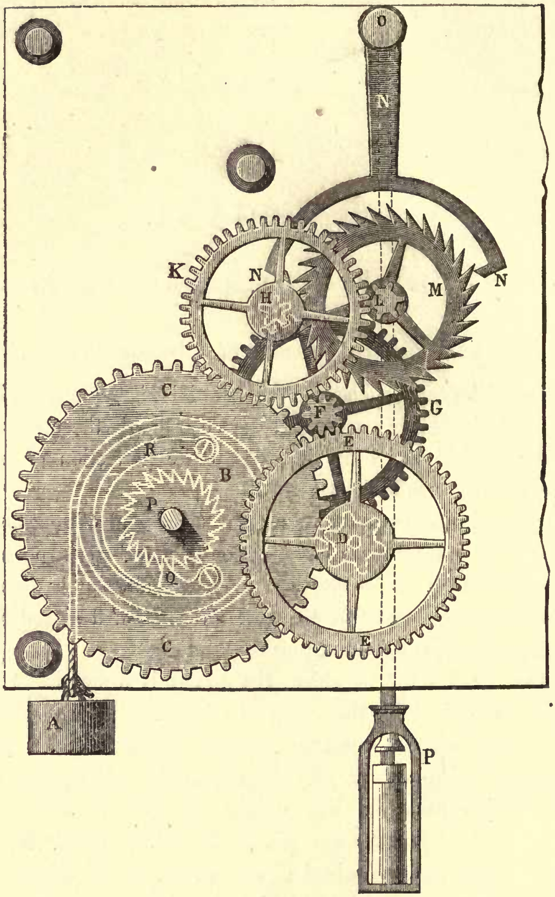
1. — The order and succession of celestial phenomena make time a most important element in astronomy, and accordingly the utmost scientific and mechanical skill has been devoted to the perfection of instruments to indicate and measure its lapse; § 37.

The best time-keepers now in use are the *Clock* and *Chronometer.* Both consist essentially of a motor, a combination of wheel-work to transmit and qualify the motion it impresses, and a check, alternately to arrest and liberate the movement, and thus to mark an *interval* designed to be some aliquot part of a day, the natural unit of duration.

2. — *The Clock.*--In the clock, the motor is a weight A suspended from a cord wound about the drum B of a wheel C, and the check is the anchor escapement N, controlled by the vibrations of a pendulum P, whose rod is geared to an arm projecting from the axis O, with which the anchor is firmly connected. The weight A turns the drum B and its wheel C; the wheel C turns the pinion D and its wheel E; the latter turns the pinion F and its wheel G, and so on to the pinion L and its wheel M, called the scape-wheel, of which the teeth are considerably under-cut, so as to turn their points in the direction of the motion. The flukes of the anchor are turned inward, forming two projections called *pallets.* The distance between the ends of the pallets is less than that between the points of two teeth that lie nearest the line drawn from one pallet to the other; and no two teeth can, therefore, pass the same pallet without the wheel being arrested by the contact of a tooth on the opposite side with the other.

With the swing of the pendulum the anchor oscillates, and one pallet is thus made to approach while the other recedes from the wheel. As soon as the receding pallet disengages itself from a tooth, the wheel is turned

Fig. 6.



by the motor and intermediate machinery till arrested by the approaching pallet, now interposed between its teeth on the opposite side. The returning swing of the pendulum reverses the pallet motion, liberates the wheel long enough for another tooth to pass, and again arrests it, and so on.

Thus, by regulating the length of the pendulum and number of teeth on the scape-wheel, an index or hand connected with the arbor of the latter may be made to travel by successive leaps, as it were, around the circumference of a circle on the dial-plate in any given time.

3. — If the anchor be connected with the *seconds pendulum*, and there be sixty teeth on the wheel, each leap will mark a second. The

motions of the minute and hour hands are regulated by suitably proportioning the relative dimensions of the intermediate wheels with whose arbors these hands are connected.

4.—The scape-wheel being in a state of constant tension by the incessant action of the motor, its teeth must act upon the pallets first by a blow and then by a pressure during the time of contact. The bearing surfaces, of which there are two on each pallet, inclined to one another, are so cut that the direction of the blow on the first from the tooth of the scape-wheel passes through the axis of the pendulum's motion, while the pressure from the same tooth on the second passes clear of that axis and accelerates the motion in the direction of the swing, thus restoring whatever of loss may come from friction and atmospheric resistance.

5. — The pendulum bob possesses the principle of compensation. It consists of a cylindrical glass vessel resting upon a plate at the end of the pendulum rod. This vessel is filled with mercury to a depth so adjusted to the length of the rod as to elevate by its expansion or depress by its contraction the centre of oscillation just as much as this centre is depressed by the expansion or elevated by the contraction of the rod during a change of temperature. The distance between the axes of suspension and of oscillation being thus made invariable, the time of vibration will continue constant, and the check be interposed at equal intervals.

6. — *Chronometer.*—The chronometer is an accurately constructed balance watch, uniting great portability with extreme accuracy. It is of various sizes, the larger having dial-plates from three to four inches in diameter, and running from two to eight days between the windings. The larger kind are suspended upon gimbals to secure uniformity of position, are mounted in boxes, and are called *box chronometers.* The smaller kind resemble in shape and size a common watch, are worn in the pocket, and are called *pocket chronometers.*

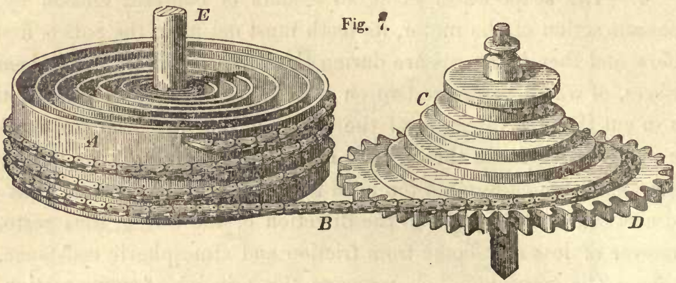
7. — The motor is an elastic spiral spring inclosed in a short cylindrical box A, called the *barrel*, one end being permanently fastened to a stationary axis E, about which the barrel freely turns, and the other to the inner surface of the barrel.

The barrel being turned in the direction of the coils of the spring, the elastic force of the latter is brought more and more into play, and its variable action thus produced is communicated by means of a chain B to a variable lever C, called a *fusee,* whose office is to modify and transmit it uniformly to the works of the instrument.

The fusee is a conical solid having its surface broken into a spiral shoulder, running from one end to the other, the curve being so regulated

that the distance of any one of its points from the axis of the fusee's motion multiplied into the force of the spring, acting through the intermedium of the chain, shall be a constant quantity; and as the main wheel D, which

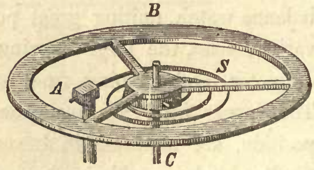
Fig. 7.



gives motion to the rest, is firmly secured to the fusee, the motion is made to act uniformly upon the instrument.

8. — The swings of the pendulum by which the check was alternately interposed between and withdrawn from the teeth of the scape-wheel in the clock, are, in the chronometer, replaced by the vibrations of what is called the balance. This consists of a wheel B, freely movable about an axis C, and a thin spiral spring S, one end of which is securely fastened to the hub of the wheel, and the other to a fixed support A. If when the spring is free from tension, the wheel be brought to rest it will remain so, just as a pendulum bob brought to rest at its lowest point will remain immovable. If from this position of the wheel it be turned in either direction about its axis, the spring will wind or unwind, the elastic force of the spring will be called into play, and will, when the wheel is unobstructed, carry it back to its position of equilibrium. But having reached this position, its living force carries it beyond; the action of the spring is reversed, and, after destroying the living force, will reverse the motion; the wheel will return to its position of equilibrium, which it will reach with a living force equal to that it had before at the same place, but in a contrary direction. The wheel will pass on, the action of the spring be reversed, the wheel will return as before, and thus the vibrations be continued forever, as in the case of the pendulum, but for the waste of living force from friction, atmospheric resistance, and absence of *perfect* elasticity in the spring.

Fig. 8.

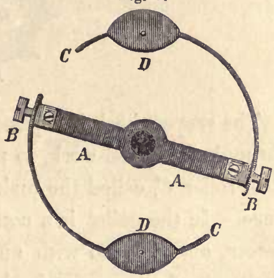


9. — The angular acceleration impressed upon the balance by the spring

is measured by the moment of its elastic force divided by the moment of inertia of the entire balance. When the temperature is increased, the spring is lengthened and the elastic force it exerts lessened; the wheel is expanded, its matter thrown further from the axis of motion, and the moment of inertia consequently increased. On both accounts the angular acceleration is diminished, and the balance will vibrate slower, and the intervals between the checks be increased. The effect is just reversed when the temperature is diminished. This is the source of greatest difficulty with all portable time-keepers, and renders the common watch worthless for any thing beyond an approximate indicator of the time.

10. — To remedy this defect, the common wheel is replaced by what is called an *expansion balance*, which is represented in the figure. A A is a bar which receives the end of the arbor into an aperture at its middle point. To the ends of the bar are securely attached two compound metallic curves C C, composed of two concentric strips, one of steel and the other of brass, the latter being on the convex side; these are soldered or burned together throughout their entire length. Each of these curved pieces carries a heavy mass D D, movable from one end to the other, but capable of being secured in any one place by means of a small clamp-screw shown in the drawing.

Fig. 9.

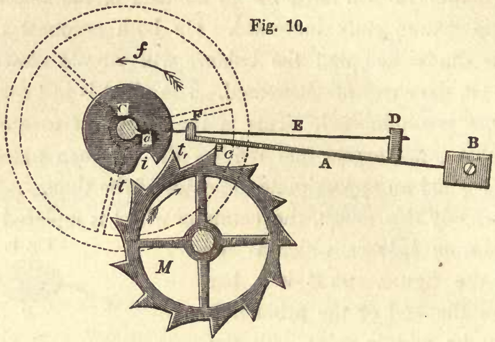


Now when the temperature increases, the exterior brass expanding more than the interior steel, the ends C C are thrown inward towards the arbor, while the ends of the bar are thrown outward, but through a much less distance; and thus by properly adjusting the places of the masses D D, the moment of inertia of the balance may be made to vary directly as the moment of the elastic force of the spring; in which case the angular acceleration becomes constant, and the intervals between the interposition of the checks equal.

11. — To regulate the rate, two large-headed screws B B, called *mean-time screws*, are inserted, one into each end of the bar. If the chronometer run too slow, the moment of inertia is too great for that of the spring, and these screws must be screwed up, which has the effect to lessen the distance of their heads from the axis of motion, and thus to lessen the moment of inertia, and increase the angular acceleration. If the chronometer run too fast, the screws must be unscrewed, the effect of which must be obvious.

12. — The *escapement* is of the kind usually called the *detached*, from the fact that except at certain instants of time, the whole appendage of the balance-spring is relieved from the action of the scape-wheel.

Fig. 10.



The scape-wheel is represented at M; it is urged by the motor, acting through the wheel-work, to move in the direction of the arrow-head. A steel roller C, called the main pallet, is firmly fixed to the arbor of the balance. In the pallet is a notch i, having one of its faces considerably undercut, and covered with an agate or ruby plate to receive the action of the teeth of the scape-wheel. Securely fixed to one of the frame-plates of the chronometer is a stud B, and to this is attached a spring A, called the *detent;* this spring is extremely thin and weak at the stud B. Attached to the detent is a stud D. A ruby pin projects from the detent at c, which receives a tooth of the scape-wheel when one escapes from the pallet bearing i. From the stud D proceeds a very delicate spring E, called the *lifting spring*, which rests upon and extends beyond a projection F from the end of the detent; this projection being so made that the lifting spring cannot move in the direction *from* the scape-wheel without taking the detent with it, and thus lifting, as it were, the pin c from the tooth with which it is in contact, while it leaves the lifting spring free to move *towards* the scape-wheel without disturbing the detent. Concentric with the main pallet, attached to and just above it, is a small projecting stud a, called the *lifting pallet*, which is flattened on the face turned from the scape-wheel and rounded on the other. The flattened is called the *lifting face.*

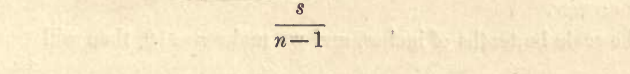
13. — *Mode of Action.*—In the position of the figure, the main pallet, under the action of the balance-spring, is moving in the direction of the arrow-head f, and the lifting pallet is coming with its lifting face in contact with the lifting spring E, which it lifts with the detent so as to raise the pin c clear of the tooth of the scape-wheel with which it is in contact.

By the time the wheel is free from the pin c, the main pallet has advanced far enough to receive an impulse from the tooth t upon its jewelled surface i, and before this tooth escapes, the lifting pallet a parts with the lifting spring E, and the detent returns to its place of rest and interposes the pin c to receive the tooth $t_{\prime}$ as soon as the tooth t has been liberated by the onward movement of the main pallet from its face i. The balance having performed a vibration by the impulse given to the main pallet, returns by the action of the balance-spring, and with it the lifting pallet a, whose rounded face, pressing against the lifting spring E, raises it and passes, first the detent without disturbing the latter, then the lifting spring, and moves on till the balance has completed the vibration, when it returns to the position indicated in the figure, and the same evolution is performed again; the balance thus making two vibrations for every impulse.

The Vernier.

1.—This is a device by which the value of any portion of the lineal distance between two divisions of a graduated scale of equal parts may be found in terms of the space itself.

It consists of a scale whose length is equal to any assumed number of parts of that to be subdivided, and is divided into equal parts of which the number is one greater or one less than the number of the primary scale taken for the length of the vernier.



Let AD be any scale of equal parts, and denote by s the length of $n-1$ of these parts; then will

$$\frac{s}{n-1}$$

be the value of the unit of the scale. Take a vernier BE of equal length s, and suppose it divided into n equal parts, then will

$$\frac{s}{n}$$

be the length of one of its parts, and the difference of length between m parts of the scale and an equal number of parts of the vernier, will be

$$\frac{ms}{n-1} - \frac{ms}{n} = \frac{m \cdot s}{n(n-1)}. \quad - \; - \; - \; - \; - \; - \; - \quad (1)$$

But $\frac{s}{n-1}$ is the value of the unit of the scale, and n the whole number of divisions of the vernier; denoting the first by V, this difference may be written

$$\frac{m}{n} V.$$

Now, the length of a part on the scale is greater than that on the vernier, and the number of parts on the vernier is greater by one than the number in an equal length on the scale; hence, if the m^{th} intermediate division of the vernier coincide with any one division on the scale, the zero of the vernier will fall between two divisions of the scale, and be in advance of that bearing the smaller figure by the distance expressed above; so that, taking the zero of the vernier as the index or pointer, its distance from the zero of the scale will be the number of units denoted by the figure on the division next preceding, plus the $\frac{m}{n}$th part of the unit of the scale. Thus, in the figure, A being the zero of the scale, B that of the vernier and therefore the pointer, the distance of the latter from the former will be $Aa+aB$; and because $n=10$, and the division b of the scale coincides with the 4th of the vernier, $m=4$, and the distance $AB=Aa+\frac{4}{10}.fe$.

2. — The least value that may be read with certainty is obtained by making $m=1$, which will give,

$$\frac{V}{n}.$$

Whence we have this rule for finding the lowest reading by means of the vernier, viz.:

Divide the lowest count, or unit of the scale, by the number of divisions on the vernier.

If the scale be tenths of inches, and we make $n=10$, then will

$$\frac{V}{n}=\frac{1}{10}\div 10=\frac{1}{100};$$

in which case the subdivisions will be carried to hundredths of inches.

3. — The vernier is equally applicable to all kinds of scales, to circular as well as rectilinear: the only condition being that the different parts shall be equal.

Suppose each degree on the circumference of a circle is divided into 6 equal parts, and that the number of parts on the vernier is 60, then will

$$V=10'=600''$$

and

$$\frac{V}{n}=\frac{600''}{60}=10''.$$

So that the reading of angles with an instrument having such a circle may be carried to ten seconds.

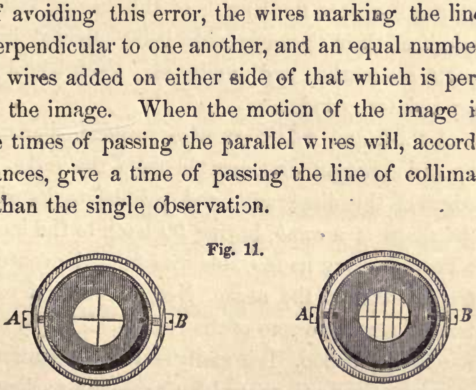
Micrometer.

1.—The Micrometer is an instrument employed to make minute measurements, and is applicable alike to time and linear distance. It has various forms.

1*.—*The Reticle.*—He who views a distant object through a telescope, does not look at the object but at its image within the tube of the instrument. The image of a point is always in a plane through the focus of the lens conjugate to the point itself, and perpendicular to the tube of the telescope. The visible portion of this plane is called the *field of view.* Some point in the field of view is arbitrarily assumed as an origin of reference, and marked by the intersection of a pair of cross wires. The line through this point and the optical centre of the field lens, is called the *line of collimation.*

2.—If the telescope be at rest and an object in motion, the image of any one of its points will when visible pass across the field of view; and one of the opaque wires being made to coincide with its path, the image will move directly towards the line of collimation, and the exact instant of its reaching it may be noted. But every such observation is liable to error. To increase the chances of avoiding this error, the wires marking the line of collimation are made perpendicular to one another, and an equal number of equidistant and parallel wires added on either side of that which is perpendicular to the path of the image. When the motion of the image is uniform, an average of the times of passing the parallel wires will, according to the doctrine of chances, give a time of passing the line of collimation more free from error than the single observation.

Fig. 11.



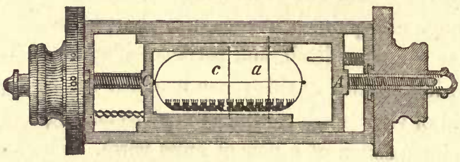
3.—This simple form of the micrometer is called a *reticle.* The wires or spider lines are stretched across a circular metallic diaphragm pierced by a large concentric opening. On the edge of the diaphragm, and in the prolongation of the single wire, two studs project at right angles to its plane; and these, with two antagonistic screws $A\,B$, hold the reticle in po-

sition; the screws, for this purpose, passing through the tube of the telescope and leaving the heads exposed for purposes of adjustment.

4. — *Position Filar Micrometer.*—The purpose of this instrument is to measure the angles at the observer, subtended by the distances between objects that appear very close together, and to determine the positions of the planes of these angles. It consists of two parts, viz.: One to measure the angle between the objects; and the other, the inclination of the plane of the objects and observer to some co-ordinate plane.

5. — The first is represented in the figure. a and c are two fine parallel wires, which are made to move at right angles to their lengths by means of screws firmly connected with the forks A and C, to whose prongs they are attached. The screws have fifty threads to the inch, and are

Fig. 12.



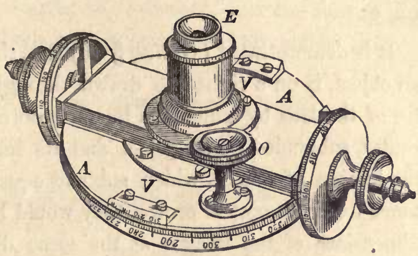
moved by nuts so mounted as to admit of a motion of rotation without translation, so that by turning the nuts a motion of translation is communicated to the wires in either direction, depending upon the direction of the rotation. The outer surfaces of the nuts are cylindrical, and enter friction tight the central perforations of two circular wheels whose planes are perpendicular to the lengths of the screws, and which are large enough to admit of their circumferences being divided into 100 equal parts, which parts are marked and numbered. Each wheel is provided with a stationary pointer or index.

A third and stationary wire, perpendicular to the first two, is supported by a diaphragm disconnected from the forks. Upon one of the interior edges of this diaphragm, and parallel to its wire, is a graduated scale in the shape of a *comb*, having 50 teeth to the inch, so that one revolution of a nut will carry its movable wire from the centre of one valley between the teeth to that of the next. Near the central valley of the scale is a small hole to mark the zero of the comb-scale, from which the scale is estimated in either direction. It is easily seen that a turn of the nut-head through one of its divisions will move its wire through a linear distance equal to $\frac{1}{100}$ of $\frac{1}{50}$ or $\frac{1}{5000}$ of an inch; and having ascertained by the measurement of some small distance on the circumference of a great circle of the celestial sphere, or by the process in Example, p. 249, its equivalent in arc, this, the

micrometer part of the arrangement, is readily applied to the determination of small angles.

6. — The second and position part consists of a circular plate *A A*, called the *position circle*, some three or four inches in diameter, having its circumference divided into 360°, which are again subdivided to any convenient extent. The central part is cut away, and the micrometer arrangement so attached, with its wires parallel to the position circle, as to admit of a free motion of rotation about an axis through its centre, and perpendicular to the plane of the wires. To the revolving plate of the micrometer part are attached two verniers *V V*, and motion is communicated to the latter by a ratchet and pinion, of which latter the head is seen at *O*. The microscope by which the wires and comb-scale are magnified, and which serves also for the eye-glass of the telescope, is represented at *E*. By means of a screw cut upon a projecting ring around the large and central aperture of the position circle, the instrument, as represented in the figure, is attached to the tail end of the telescope.

Fig. 18.



7. — To measure the angular distance between two objects in the field of view, turn the head *O* till the fixed wire passes through their images, then bisect the images by the movable wires; note the reading on the comb-scale and upon the heads; take their sum or difference according as the wires are on opposite sides, or same side of the zero of the comb-scale. This reduced to arc will be the measure sought. Note also the reading of the position circle; this will give the inclination of the plane of the angle to the plane through the zero of the position circle. A second angle being measured in the same way, the difference between the second and first reading of the position circle will give the inclination of the planes of the two angles.

Micrometer Revolution.

The micrometer being supposed in place, and the eye-piece pressed forward far enough to obtain a distinct view of the wires, the telescope is directed to some distant object, and adjusted to distinct vision. An image of the object will be formed on the plane of the wires, and any one of its

linear dimensions may be measured by turning the position circle till the stationary wire coincides with, and the movable wires pass through the extremities of its image. The number of *entire* comb-teeth between the movable wires, multiplied by 100, and this product increased by the sum of the readings of the screw-heads, will give the linear dimensions of the image expressed in units of the screw-head. The value of the latter is, in the case we have taken, $\frac{1}{5000}$ of an inch. To find the angle subtended by the object, we must know the *angular* value of the unit on the screw-head.

It is demonstrated (*Optics*, § 60) that the optical image of any point of an object, is on a right line drawn through the point and the optical centre of the lens by which the image is formed. The angles, at the optical centre, subtended by an object and its image, are therefore equal, and if the images of objects which subtend equal angles were at the same distance from the optical centre, they would be of the same size. The linear dimensions of the images at the same distance from the optical centre, would therefore be proportional to the angles subtended by their respective objects, and to find the angular value in question, it would be sufficient *to cause the image of some well-defined object, whose distance and dimensions are known, to be embraced by the wires, and to divide the angle which the object subtends, expressed in seconds, as determined trigonometrically, by the number of units of the screw-heads, which indicate their separation.* But the distances and therefore the dimensions of images, whose objects subtend the same angle, are variable, being dependent on the distance of the objects, and from the value found by the above process must be deduced that which would have resulted had the image been formed at some constant distance, which is that of the principal focus.

Let f and f'' denote the distances respectively of the object and its image from the optical centre, and $F_{\prime\prime}$ the principal focal distance of the object-glass, supposed convex. Then, *Optics*, § 44, Eq. (40),

$$\frac{F_{\prime\prime}}{f''} = -\frac{f - F_{\prime\prime}}{f},$$

and denoting by n and N, the number of units of the screw-heads when the image is embraced at the distances f'' and $F_{\prime\prime}$ respectively, we shall have, *Optics*, § 64, Eq. (58),

$$f'' : F_{\prime\prime} :: n : N;$$

whence

$$N = \frac{F_{\prime\prime}}{f''} . n = -n . \frac{f - F_{\prime\prime}}{f}$$

and calling a, the number of seconds in the angle subtended by the object, we have, by the rule just given,

$$\frac{a}{N} = -\frac{a.f}{n.(f - F_{,,})} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Example.—The length of the object measured in a direction perpendicular to the line of sight was 3 feet; the distance from the object-glass, 261.9 yards; the principal focal length, 45.75 inches; and the sum of the divisions on the screw-heads indicating the separation of the wires, 1819.

Then

$$f = 261.9^{\text{yds.}};\ F_{,,} = 45.75^{\text{in.}} = 1.2708^{\text{yds.}};\ n = 1819.$$

$$f - F_{,,} = 260.6292^{\text{yds.}}.$$

$$\tan \tfrac{1}{2} a = \frac{R \,.\, 0.5^{\text{yds.}}}{261.9^{\text{yds.}}},\ \text{of which the log. is } 7.280835\,;$$

whence

$$a = 13'\ 07''.57 = 787''.57.$$

Log. a		. . . . .	2.8962892
" f		. . . .	2.4181355
" n	a comp.	. .	-4.7401673
" $f - F_{,,}$ "		. .	-3.5839923
" $\frac{a}{N} = 0''.4351$		. .	-1.6385843

Now, to measure the angle subtended by the distance between any two points, direct the telescope so as to get the images of the points in the field, and turn the micrometer till the stationary wire apparently passes through them, and by a motion of the screw-heads bring the movable wires to the images—the number of units of the screw-head, which indicate the separation of the wires, multiplied by the decimal $0''.4351$, will give the number of seconds in the angle.

The value of $\frac{a}{N}$, being a function of $F_{,,}$, Eq. (a), will of course vary with the object-glass, but is perfectly independent of the eye-glass.

If the distance f be so great that $F_{,,}$ may be neglected in comparison, then will Eq. (a) give

$$N = n,$$

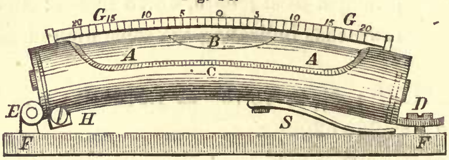
which will be the case when the angular value is determined from astronomical objects.

Spirit-Level.

1. — This is an instrument used to adjust a line to a given position in reference to the *horizon.*

It consists of a *cylindrical glass tube* $A\,A$, whose axis is the arc of a circle. This tube is filled nearly full with some one of the more perfect fluids, such as alcohol or naphthalic ether, leaving a small portion of air, seen at B, called the *air-bubble*, and hermetically sealed at both ends. It

Fig. 14.



is then usually set in a metallic tube C, very much cut away on one side from the middle towards the ends, so as to exhibit the bubble and fluid when in a horizontal position. This metallic tube is connected with a plate of metal $F\,F$, by a hinge E and screw D, the axis of the hinge being perpendicular, and that of the screw parallel to the plane of the circular axis of the level.

2. — A scale of equal parts is cut either upon the upper surface of the glass tube or upon a slip of ivory and metal lying in the plane of the tube's curve, as represented at $G\,G$. The divisions of the scale being numbered, the value of the spaces in arc is readily ascertained by attaching the level to the face of a vertical graduated circle, and turning the latter sufficiently to cause the air-bubble to pass from one end of the scale to the other. The angular space passed over by the circle reduced to seconds, divided by the number of units on the scale traversed by the bubble, will give the value of the unit in some multiple of the second.

3. — *Use.*—The surface of the fluid being always horizontal, the line connecting the ends of the bubble will be a level chord of the level's arc, and the radius passing through the point of the scale midway between the ends of the bubble will be vertical.

Now, suppose any line of an instrument with which the level is used to be made parallel either to the radius passing through the zero of the scale, or to the chord whose ends are marked by the same numbers; then, to make this line vertical in the first case, or horizontal in the second, move the instrument, the level being securely attached, till the ends of the bubble are equally distant from the zero.

If the ends of the bubble be not at the same distance from the zero, the inclination x of the line in question to the vertical or horizontal

direction is thus found: Let a denote the semi-length of the bubble, m and n the numbers of the scale at its extremities, then will

$$a+x=m,$$
$$a-x=n;$$

whence

$$x=\frac{m-n}{2}=l \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

This value of x being independent of the length of the bubble, which is indeed a variable quantity, even in the same level, because of its varying temperature, gives the inclination of the line under consideration to its proper position, when the level is adjusted to the instrument.

If the lower surface of the plate FF be parallel to the chords of equal numbers, the inclination of any given line or plane may be ascertained by laying this plate upon it and applying the above rule.

But if the lower surface of the plate be not parallel to the chords of equal numbers, its inclination to them, and that of the plane or line in question to the horizontal or vertical direction may nevertheless be found thus: Denoting the first by y, and the latter, as before, by x, and using the notation of equation (2), we have for one position of the level,

$$x=l-y,$$

and for the reversed position of the plate with its level,

$$x=l'+y,$$

whence

$$x=\frac{l+l'}{2}=\frac{\overline{m-n}+\overline{m'-n'}}{4},$$

$$y=\frac{l-l'}{2}=\frac{\overline{m-n}-\overline{m'-n'}}{4}.$$

If the given surface or line be provided with adjusting screws, as is the case in all astronomical instruments, the ends of the bubble may be brought to the same reading in the first position of the level, in which case, we have $m=n$, and

$$x=\frac{m'-n'}{4}=-y \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

The angle y is called the error of the level, and the angle x the error in level of the instrument, and the above equation gives this rule for finding and correcting these errors, viz.:

The level being placed over the given line, bring, by means of the adjusting screws of the instrument, the bubble to read the same at both ends; then reverse the level, or turn it end for end, and take one fourth of the difference of the new readings; add this to the lesser of the readings, and turn the screw D till the end of the bubble nearest the zero reach the number answering to this sum, to which add again the same quantity, and bring the end of the bubble to this new reading by the adjusting screws of the instrument. The ends of the bubble will stand at the same numbers, and both errors will be destroyed.

Reading Microscope.

1. — This instrument, like the vernier, has for its object to read and subdivide the space between two consecutive divisions of any scale of equal parts, and is the most perfect yet devised for this purpose.

It is a compound microscope, whose object-glass forms an enlarged image of the space to be divided. This image is thrown upon the plane of two spider-lines or wires, arranged in the form of a St. Andrew's cross, and so placed that a line bisecting its smaller angles is parallel to the cuts or division marks of the scale. The cross is attached to a diaphragm, which is moved by a micrometer screw in the direction of its plane, perpendicular to the axis of the microscope. The head of the screw is divided into any number of equal parts, depending upon the nature of the scale and the extent to which the subdivisions are to be carried. The numbers on the head are so placed that when the screw is turned in the direction to bring them in the order of their increase to a fixed pointer, the cross shall move along the image-scale in the direction in which its numbers decrease.

Within the barrel of the microscope is a stationary comb-scale, like that in the position micrometer. Its plane is parallel to that of the cross, and the distance between the centres of two valleys, separated by a single tooth, is equal to the space over which the cross is moved by a single revolution of the screw. Every fifth valley is cut deeper than the others to facilitate the reading; and near the bottom of the central valley of the comb is a small circular aperture, to mark the zero position of the pointer or index, which is a small wire attached to the movable diaphragm, and so placed that its prolongation shall bisect the smaller angles of the cross.

In (1), $A\,A$ is the main tube of the microscope, passing through a collar or support B, where it is firmly held by two milled nuts $g\,g$, which act upon a screw cut upon the outer surface of the tube. These nuts also serve to change the distance of the whole microscope from the scale to

be read ; h is the object-glass placed in a smaller tube, upon whose outer surface is also a screw, by which this glass may be moved independently

Fig. 15. 1.

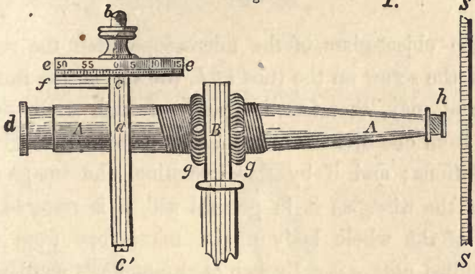
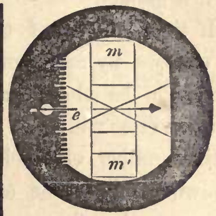


Fig. 16. 2.



of the main tube ; the diaphragm of the cross is in a working box, whose edge is seen at a ; e is the graduated head, firmly attached by a friction clamp to the nut b of the micrometer screw ; f is a pointer attached to the working box ; d is the eye-glass, which moves freely in the direction of the axis of the microscope by a sliding tube ; at c' is represented the head of a small screw, which supports and gives motion to the comb-scale within the working box, and $S\,S$ represents the edge of the scale to be subdivided. In (2) is represented the field of view, as seen when the eye is applied at d, in which $m\,m'$ is the image of the scale, with one of its cuts bisecting the smaller angles of the cross, and e the wire index at its zero position, as indicated by its being seen through the centre of the circular aperture of the comb. In this position of the pointer, the zero of the graduated head e is brought to the index f, by holding the nut b firmly in the hand, and turning the head, which is only held in its place, as before stated, by the action of the friction nut.

2. — The quotient arising from dividing the length of the image space by that over which the wires move in one revolution of the screw-head, as given by the comb-scale and head, is called *the run of the micrometer.* For convenience, the run should be an entire number.

3. — The image-scale must be accurately in the plane of the wires, otherwise there would be a parallactic motion, which would shift the position of the wires on the image-scale at every change in the position of the eye, and thus vitiate the measurement. This parallactic motion is easily detected by slightly shifting the position of the eye when looking through the eye-glass.

There are, then, two adjustments for the reading microscope, viz., that for the *run* and that for *parallax*

4. — The size of the image of an object, and its distance from the lens by which it is formed, are dependent upon the distance of the object from the lens, being greater in proportion as this distance is less, and less as it is greater.

If the distance of the object-glass of the microscope from the scale be changed by means of the screw on the tube at h, the size of the image space will be altered, and may, therefore, be made of such dimensions that the cross will move from one division to the next in order, by a given entire number of revolutions; and if by this operation, the image be thrown off the plane of the wires, as it in general will, it is restored by changing the distance of the whole body of the microscope from the scale by means of the milled nuts $g\,g$. By two or three efforts cautiously conducted, the adjustments may be made without difficulty.

To illustrate, let the scale be that of the sexagesimal division of the circle, and suppose each degree divided into twelve equal parts, each space will be equal to five minutes; if we make the run five, each tooth on the comb will be equal to one minute, and if the screw-head be divided into sixty equal parts, each of its spaces will be equal to one second; so that the circle may be read to seconds.

Now suppose on examining the run, which is done by turning the screw-head till the cross moves from one division to the next in order, it be found $5' \; 10''$; it is too great. Move the object-glass h from the plane of the circle by screwing in its tube, the image will decrease, and, if it were before on the plane of the wires, it will now pass to some position between that plane and the object-glass h. Move the whole body of the microscope by means of the milled nuts $g\,g$ towards the circle; the image will be restored to its proper position, with less dimensions than it had before. By one or two repetitions of this process the adjustments are made.

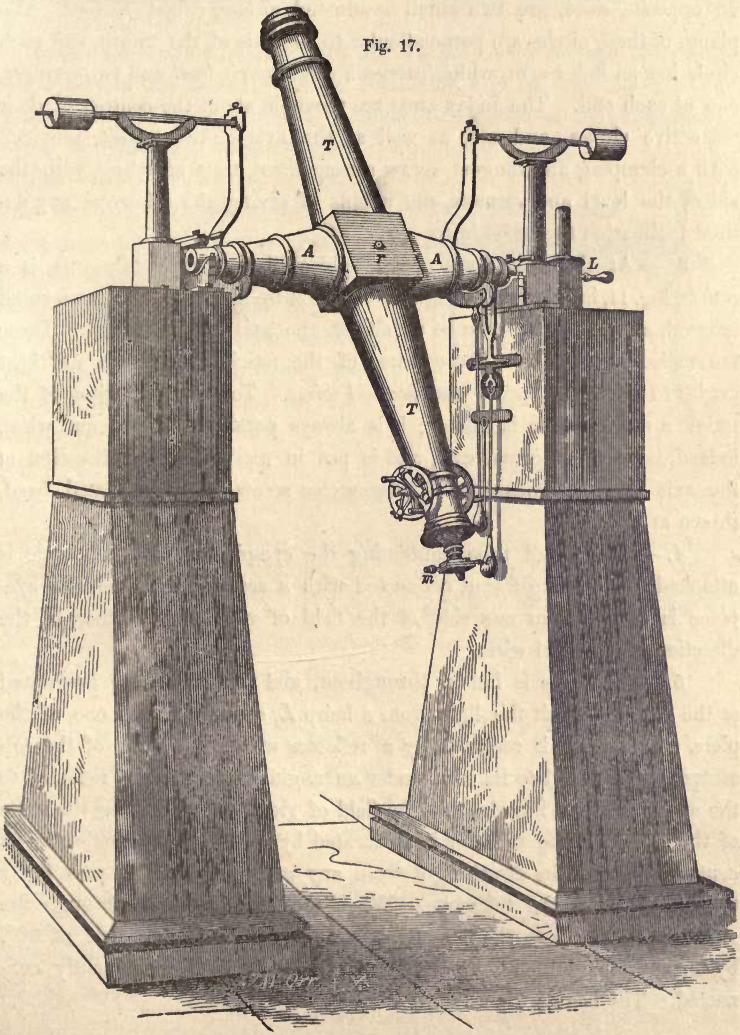
5. — The wire pointer at its zero position on the comb-scale is the index of the circle or instrument scale. When the pointer, in this position, is immediately opposite a division mark of the circle scale, say the third after that marked 27°, which is indicated by the angles of the cross being bisected by the image of that division mark, the reading is $27° \; 15' \; 00''$; but if the intersection of the cross wires falls between the third and fourth divisions after that marked 27°, then will the reading be greater than that above by the value of the distance from the cross wires to the division mark to which the cross will move by turning the screw-head in the order of its increasing numbers. To find this value, turn the screw-head in the direction just indicated till the angles of the cross are

bisected by the division mark in question, and count the *entire number* of comb teeth between the aperture and pointer, then note the reading on the screw-head; suppose the former to be 3 and the latter 41, the true reading will be 27° 18′ 41″.

The Transit.

1. — The transit is an instrument which is used in connection with a time-piece to ascertain the precise instant of a body's passing the me-

Fig. 17.



ridian of a place. It consists of a telescope TT, usually of considerable power, permanently fixed to a substantial axis AA, at right angles to its length. The axis terminates at each end in a steel pivot, accurately turned with a diamond point, to a cylindrical shape. The pivots are of equal diameters, received into notches cut in two blocks of metal, called Ys, which rest in metallic boxes, the latter being imbedded in metallic or stone piers, according as the instrument is intended to be portable or fixed.

2. — Permanently attached to the tail or eye end of the telescope, on opposite sides, are two small graduated circles, called *finders.* The planes of these circles are perpendicular to the axis of the transit, and each circle has an index-arm, which carries a small *spirit-level* and two *verniers*, one at each end. The index-arms are movable about the centres of their respective circles, and are, as well as the axis of the transit, provided with a clamping and tangent screw arrangement, thus affording, with the aid of the level and verniers, the means of giving the telescope any desired inclination to the horizon.

3. — At the solar focus of the object-glass of the telescope is a *reticle*, Fig. 11, in which the single is replaced by a double wire, with small interval, and so placed as to be parallel to the axis of the transit. These are called *axis wires.* Those wires of the reticle which are at right angles to these are called the *normal wires.* To the fixed wires of the reticle a movable one is added; it is always parallel to the normal wires, indeed, is itself a normal wire, and is put in motion in the direction of the axis wires by means of a micrometer screw, with graduated head, shown at *m.*

4. — The small tube containing the eye-piece of the telescope is attached to a sliding-frame, connected with a screw *e*, by which the eye-piece is carried from one side of the field of view to the other, in the direction of the axial wires.

5. — The axis is hollow throughout, and the pivots are perforated at the ends to admit the light from a lamp *L*, supported upon one of the piers. This light is received by a reflector within the tube of the telescope, and inclined to its axis under an angle of 45°, and is reflected to the eye-glass, thus illuminating the field of view, and exhibiting the wires of the reticle. The reflector is perforated by an elliptical opening in its centre, to permit the direct light from any external object to pass freely to the eye end of the telescope. When the illumination is through the other end of the axis, the reflector is revolved through an angle of 90°, by means of a milled-headed wire, with which it is permanently connected. The head is shown at *r.*

Fig. 18.

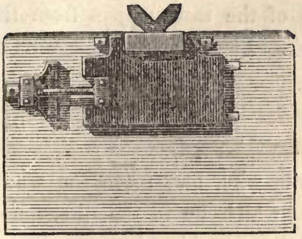
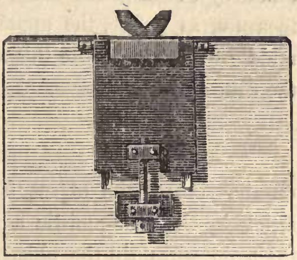
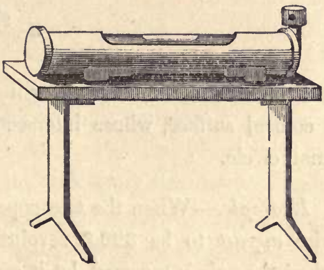


Fig. 19.



6. — The boxes which support the Ys are large enough to permit a slight play in the latter; one in a horizontal, Fig. 18, and the other in a vertical direction, Fig. 19, the motions being effected by antagonistic screws. By the first of these motions, the line of collimation is brought to the meridian, after the rougher approximations to that plane are made by other means, and by the second the axis is made horizontal by the aid of a large and delicate spirit-level, Fig. 20, mounted upon inverted Ys, far enough apart to rest upon the pivots.

Fig. 20.



Adjustments.

7. — The transit is adjusted within itself when its line of collimation is perpendicular to its axis; and it is in position, when its axis is perpendicular to the meridian. Its finders are adjusted, if the air-bubbles at their levels indicate the same reading at both ends, when the verniers indicate the true inclination of the line of collimation to the vertical or horizon.

8. — It is by no means necessary, or even desirable, to aim at perfect adjustment. It will, in general, be much safer to reduce the errors of adjustment to narrow limits, then to determine their amount, and eliminate their effect from observation, in the manner to be described presently.

9. — *Line of Collimation.*—Direct the telescope to some small, distant, and well-defined terrestrial object. Bring it apparently between the horizontal wires, and measure its distance from the central normal wire by means of the micrometer and movable wire; denote this dis-

tance by c'. Lift the transit from its Ys, turn the axis end for end, and measure, as before, the apparent distance of the same object from the middle wire, and denote this distance by c''. Place the movable wire at the distance, of

$$\frac{c'+c''}{2},$$

on the side of the object from the middle wire, and move the whole reticle by the antagonistic adjusting screws, which lie in the direction of the axial wires, till the object appears on the movable wire; the line of collimation will be adjusted.

10. — *Error of this adjustment.*—If n denote the value in arc of the micrometer's unit, then will the angle which the line of collimation makes with its proper position, before moving the diaphragm, be

$$n \cdot \frac{c'-c''}{2} = c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and the line of collimation will describe, when the telescope is moved, a conical surface, whose intersection with the celestial sphere will be a small circle.

Example.—When the telescope is pointing to the south, let the middle wire *appear* to be 326.3 revolutions to the right hand of the object; when the axis is reversed, let it appear 318.7 to the right, then will

$$n \cdot \frac{326.3-318.7}{2} = c = 3.8n,$$

and if one revolution of the micrometer correspond to the space an equatorial star would pass over in three seconds of time, then will

$$n = \frac{3s. \times 15}{100} = 0''.45,$$

and

$$c = 3.8 \times 0''.45 = 1''.71.$$

11. — *The axis.*—This must first be levelled, then moved in azimuth till it is perpendicular to the meridian.

Mount the level with its inverted Ys upon the pivots, bring the bubble to the same reading at each end by the adjusting screw of the level; reverse the level, and bring the bubble again to the same readings—half by the screw of the level and half by the vertical antagonist screws of the Y, which admits of vertical motion. Repeat the operation once or twice, and the thing is done

12.—*The error in this adjustment.*—After the first approximation, denote by e', e'', &c., the reading of the east end of the level; by w', w'', &c., the same of the west end, and let the parenthesis denote the end of the axis marked by some peculiarity, such as the clamp, or illumination; then mounting the level in its place, and writing its readings in any one position upon the same horizontal line, we may have

First position of level	e'	(w')
Level reversed	e''	(w'')
Half sums of	$\frac{\overline{e'+e''}}{2}$ . . .	$\frac{\overline{(w')+(w'')}}{2}$

These half sums are the readings which the level would have indicated in both positions had it been in perfect adjustment, and

$$\frac{\overline{(w')+(w'')}-\overline{e'+e''}}{4}=s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the error, or inclination of the axis to the horizon, expressed in the level's unit, provided its pivots be of the same size. But lest there may be a difference in the pivots, reverse the axis, and apply the level as before, and we may have

For first position of level . .	(e''') . . .	w'''
Level reversed	(e'''') . . .	w''''
Half sums	$\frac{\overline{(e''')+(e'''')}}{2}$ . .	$\frac{\overline{w'''+w''''}}{2}$

whence

$$\frac{\overline{w'''+w''''}-\overline{(e''')+(e'''')}}{4}=s' \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$\frac{s-s'}{4}=\frac{\overline{(w')+(w'')+(e''')+(e'''')}-\overline{w'''+w''''+e'+e''}}{16}=t \quad . \ (7)$$

will be the angle which the axis makes with the line whose inclination is given in equation (5), whence, denoting the inclination of the axis to the horizon, or *the angle which a plane perpendicular to the axis makes with a vertical plane at right angles to the projection of the axis, on the horizon, by l'*, we shall have

$$l'=s\pm t.$$

This value is expressed in terms of the level's unit; if n' denote the

value of this unit in seconds, we shall have, representing the angle in seconds of arc by l,

$$l=n'\,l'=n'\,(s\pm t)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The value of t for the same axis is constant, and must be determined by taking a mean of a great many careful observations. If it be positive, the pivot at the clamp end of the axis is the larger, but if negative, it is the smaller.

When the half sum of the readings on the west end is greater than that on the east, the inclination is counted *positive*, and the plane perpendicular to the axis will fall to the east of the zenith; and as it is obvious that the axis will be depressed on the side of the greater pivot, when the level indicates perfect adjustment, the upper sign, in equation (8), must be taken when that pivot is to the east, and lower when to the west.

Example.—Performing the operations indicated, let the following be the record, viz.:

First position of level . . .	71.40	(87.60)
Level reversed	78.60	(80.10)
	150.00	167.70
	167.70	
	4)17.70(4.425 $=s$	

Axis reversed.

First position of level . . .	(73.95)	84.90
Level reversed	(81.30)	77.85
	155.25	162.75
	162.75	
	4)7.50(1.875 $=s'$	

Adding the indications of the level diagonally, we have

$$\frac{(322.95)-312.75}{16}=0.6375=t.$$

Applying the level to the face of some vertical graduated circle, § 95, let 23.5 of its units correspond to 30″ then will

$$n'=\frac{30''}{23.5}=1''.276.$$

Whence for

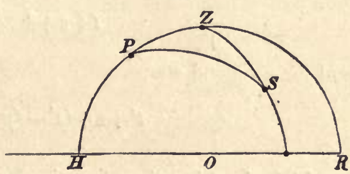
Clamp end west $l=(4.425-0.637)\times 1''.276=4''.83348$

Clamp end east $l=(1.875+0.637)\times 1''.276=3''.205312$

13.—*Azimuth adjustment.*—It is now supposed that the errors of collimation and of level are destroyed. By a reference to a map of the stars it will be seen that a straight line drawn from the Pole star to a point midway between the fifth and sixth stars, called ε and ζ respectively, in the constellation of the Great Bear, will pass sensibly over the pole. About the time when this line assumes a vertical position, direct the telescope to the Pole star, and keep its image on the middle normal wire by a motion of the horizontal adjusting screws of the Y, or by the motion of the Ys themselves, if the requisite range be beyond that of the screws, and at the instant when it is inferred from a suspended plummet, that the line referred to is exactly vertical, arrest the motion and secure the Ys. The adjustment will be sufficient for the first approximation.

Next find the amount of azimuth error. The axis being horizontal, and the line of collimation perpendicular to the axis, it is plain that in the motion of the telescope the line of collimation will describe the plane of a vertical circle, and that the angle made by this plane with the meridian is the error in question.

Fig. 21.



Let $H\,O\,R$ be the horizon, $R\,Z\,H$ the meridian, P the pole, Z the zenith, and S the star when on the line of collimation. Make,

λ = latitude of place $=90°-Z\,P$;

δ = declination of star $=90°-P\,S$, positive when north, negative when south;

$P=Z\,P\,S=$ hour angle of star;

$Z=H\,Z\,S=$ azimuth of star's position, and equal to the error sought when east.

$z=Z\,S=$ zenith distance of star.

Then, in the triangle $Z\,P\,S$,

$$\sin P=\frac{\sin Z\,.\,\sin z}{\cos\delta},$$

and because the sines of P and Z are very small,

$$P=\frac{\sin(\lambda-\delta)}{\cos\delta}.Z \quad . \quad . \quad . \quad . \qquad . \quad (9)$$

in which P and Z are expressed in seconds of arc.

Divide both members by 15 and make

$$\frac{\sin(\lambda-\delta)}{\cos\delta}=k;$$

and equation (9) becomes

$$\frac{P}{15}=\frac{k\,.\,Z}{15}\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (10)$$

in which $\frac{P}{15}$ is the time required for the star to pass from the vertical described by the line of collimation to the meridian, and if t denote the time indicated by a timepiece at the instant the star is on the central normal wire, the time of meridian passage will be

$$t+k\,.\,\frac{Z}{15}=T\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (11)$$

Let e be the error of the timepiece at the time t referred to the vernal equinox; m the *rate* or quantity by which this error is increased or diminished in one day or twenty-four hours; then, if R denote the right ascension of the star, supposed known, will

$$t+e+k\,.\,\frac{Z}{15}=R\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (12)$$

and for a second star

$$t'+e+(t'-t)m+k'\,.\,\frac{Z}{15}=R',$$

in which $t'-t$ is reduced to the decimal part of a day. Subtracting the first from the second, we get

$$t'-t\pm(t'-t)m+(k'-k)\,.\,\frac{Z}{15}=R'-R,$$

in which the upper sign is used when the timepiece runs too slow, and the lower when too fast; whence,

$$\frac{Z}{15}=\frac{\overline{R'-R-(t'-t)}\pm\overline{(t'-t)\,m}}{k'-k}\quad .\ .\ .\ .\ .\quad (13)$$

Z is hence known, and for which the instrument may be corrected, if desired. This value in equation (11), gives the time of meridian passage, and in equation (12), which may be written

$$e=R-t-k\,.\,\frac{Z}{15}=R-T,\quad .\ .\ .\ .\ .\ .\ .\quad (13')$$

gives the error of the timepiece.

The sign of the quantity k changes when the declination of the star exceeds the latitude, and also when the star passes below the pole, since in this latter case δ becomes 90°, plus the polar distance. The right ascension for all stars which pass below the pole must be diminished by twelve hours.

Using the Polar star in its upper and lower passage instead of two separate stars, equation (13) becomes

$$\frac{Z}{15}=\frac{12^{\text{h}}-\overline{(t'-t)\pm(t'-t)}m}{k'+k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

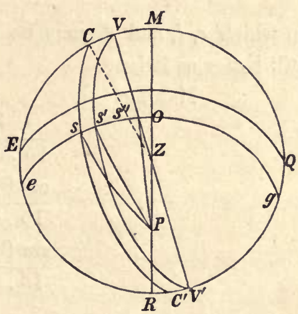
When three consecutive transits of the Pole star are observed, and the intervals are equal, Z will be zero, and the transit's axis is perpendicular to the meridian.

The values of k and k', in equation (13), must be found from stars differing at least 50° in declination.

14. — Let it now be supposed that after adjusting the transit in the manner explained, there is still (as in general there will be) remaining a small error in collimation, level, and azimuth. It remains to be shown how the effects of these may be eliminated from the observation, and a result obtained the same as though the instrument had been perfect.

Let all the circles referred to in what precedes be projected on the horizon, represented in $MERQ$. Let Z be the zenith; P the pole; MZR the meridian; VZV' a vertical circle at right angles to the projection of the axis of the transit; $Vs'V'$ the circle at right angles to the axis; CsC' the parallel small circle cut from the celestial sphere by the motion of the line of collimation; EQ the equator; and esg the diurnal path of a star.

Fig. 22.



When the star appears on the central normal wire, it will be at s; and if the time be noted and increased by the angle sPO, expressed in time, we shall have the indication of the timekeeper when the star is on the meridian. Now,

$$sPO=sPs'+s'Ps''+s''PO;$$

the angle sPs' is measured by an arc of the equator, which is equal to

$s\,s'$ divided by the cosine of the distance of $s\,s'$ from the equator, which distance is the declination of the star. But

$$s\,s'=C\,V=c,$$

c being as before the error of collimation; hence,

$$s\,P\,s'=\frac{c}{\cos\,\delta}.$$

The angle $s'Vs''$ is the error of the level denoted by l. Then regarding $P\,s''\,V$ as the arc of a great circle, from which it will differ by an inappreciable quantity within the limits of the supposed errors, we shall have, in the triangle $P\,s'\,V$, writing the small angles for their sines,

$$s'P\,s''=l\,.\,\frac{\sin\,Vs'}{\sin\,P\,s'}=l\,.\,\frac{\cos\,(\lambda-\delta)}{\cos\,\delta},$$

representing the zenith distance by $(\lambda-\delta)$, to which it is nearly equal, and regarding Vs' as the altitude, from which it differs but by a very small quantity.

The angle $s''P\,O$ is given by equation (9), Z denoting as before the azimuth error. Whence, denoting by t the time of observation, we obtain for the time of meridian passage

$$t+\frac{c}{15\,.\cos\,\delta}+\frac{l}{15}\,.\,\frac{\cos\,(\lambda-\delta)}{\cos\,\delta}+\frac{Z}{15}\,.\,\frac{\sin\,(\lambda-\delta)}{\cos\,\delta}=T\quad.\quad.\quad(15)$$

in which c, l, and Z may be found in the manner already indicated, or still better as follows.

Making

$$\left.\begin{aligned}\frac{1}{15\,.\cos\,\delta}&=C\\ \frac{\cos\,(\lambda-\delta)}{15\,.\cos\,\delta}&=L\\ \frac{\sin\,(\lambda-\delta)}{15\,.\cos\,\delta}&=Z_1\end{aligned}\right\}\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(16)$$

and supposing the timepiece regulated by the vernal equinox, and representing its error at the time t by e, and denoting by R, the right ascension of the star, we obtain

$$t+e+c\,.\,C+l\,.\,L+Z\,.\,Z_1=R\quad.\quad.\quad.\quad.\quad.\quad(17)$$

in which, if e, c, l, and Z be regarded as unknown, their values may be found by carefully observing four stars, whose positions are well known, and which differ but little in right ascension, and considerably in declina-

tion The values of C, L, and Z_1 being computed in each case from equation (16), we may have

$$\begin{aligned}
t' &+e'+c\,.\,C' +l\,.\,L' +z\,.\,Z_1' =R' \\
t'' &+e'+c\,.\,C'' +l\,.\,L'' +z\,.\,Z_1'' =R'' \\
t''' &+e'+c\,.\,C''' +l\,.\,L''' +z\,.\,Z_1''' =R''' \\
t'''' &+e'+c\,.\,C'''' +l\,.\,L'''' +z\,.\,Z_1'''' =R''''
\end{aligned}$$

which are sufficient. But as there are always slight errors in the observations themselves, it would be well, where great accuracy is required, to increase the number of these equations, and treat them after the method of least squares.

15. — *The finding circles.*—These may indicate zenith distances, altitudes, or polar distances. The rule for adjusting is the same for all.

Direct the telescope to the distant horizon, and move it till the image of some small object appear midway between the double axial wires: clamp the axis, move the index-arm till its level indicates the same reading at both ends of the bubble, and note the reading of the vernier. Unclamp and reverse the axis; bring the image of the same object again between the same wires, and clamp the axis; move the index-arm till the bubble has the same reading at each end, and again note the reading of the vernier. If the vernier reading be the same as before, the circles are in adjustment; if not, add the readings together, take the half sum, move the index-arm till the vernier is brought to the reading indicated by this half sum, clamp the index-arm, and bring the air-bubble so as to have the same reading at each end by the adjusting screws of the level. It would be well to verify by repeating the process. It may be, that the finders are graduated from 0° to 360°, in which case, if the first reading were $a°$, the second ought to be $360°-a°$.

16. —The adjustments in azimuth, collimation, and level being *perfected*, the middle normal wire will be a visible representation of that portion of the celestial meridian to which the telescope is pointed; and when a star is seen to cross this wire in the telescope, it is in the act of culminating. The precise instant of this event being noted by the clock or chronometer, the time of meridian passage is known, and any error in noting this precise time is lessened by the use of the lateral wires of the reticle, as already explained.

17. — Besides, these lateral wires increase the chances of securing an observation that might, without them, be lost. It frequently happens that efforts to obtain the time of a body's passing the middle or other wire are defeated by the presence of clouds, or other accidental circumstances,

in which, if the time of passing any one be obtained, that of passing the middle or mean place of the wires, when not equally distant, may be deduced thus.

Let t_1, t_2, t_3, &c., be the times of crossing the several wires in order, then will

$$\frac{t_1+t_2+t_3+\ldots t_n}{n}=t_m, \quad \ldots\ldots\ldots \quad (18)$$

in which t_m denotes the time of the body's crossing the mean position of the wires, and n the number of wires. And

$$\left.\begin{array}{l}(t_m-t_1)\,.\cos\delta=i_1,\\(t_m-t_2)\,.\cos\delta=i_2,\\(t_m-t_3)\,.\cos\delta=i_3,\\ \quad .\quad .\quad .\quad .\quad =.\\(t_m-t_n)\,.\cos\delta=i_n.\end{array}\right\} \quad \ldots\ldots\ldots \quad (19)$$

in which δ denotes the declination of the body observed, and i_1, i_2, $i_3 \ldots i_n$, the constant intervals of time required for a body in the equator to pass over the distances which separate the several wires from their mean position.

Adding equations (19) together, we obtain

$$t_m=\frac{\Sigma t}{n}+\frac{\Sigma i}{n\cos\delta} \quad \ldots\ldots\ldots \quad (20)$$

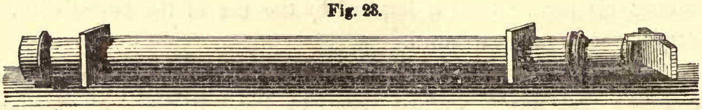
in which Σ denotes the algebraic sum of the quantities expressed by the letter written after it.

By carefully observing a star whose declination is known, we obtain the values of i_1, i_2, &c.; and these being tabulated with their proper signs, equation (20) will give the time of a body's passing the mean position from the time of passing one or more of the threads.

The Collimating Telescope.

1.—In some situations it would not be possible to obtain a distant mark by which to collimate, and a near one could not be used in consequence of its image falling too far behind the reticle. In such cases recourse must be had to what is called the collimating telescope.

Fig. 28.



This is a telescope whose eye-piece is removed, and upon its tube is mounted a small swing-frame, supporting a reflector, by means of which

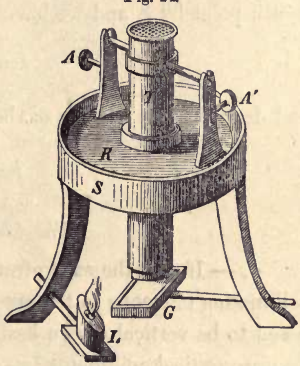
sufficient light may be thrown through the telescope to illuminate a pair of cross wires, situated at the solar focus of the object-glass.

In this position of the wires, we have, from the principles of optics, these facts, viz.: the rays composing the pencil of light proceeding from any point of the cross, will emerge from the collimator parallel to a line drawn through that point and the optical centre of the lens; and if the telescope of the transit be directed towards the collimator so as to receive these rays, an image of the point in question will appear in its solar focus, and on a line drawn through the optical centre of its object-glass, parallel to these same rays.

The Vertical Collimator.

1. — This instrument is used for the double purpose of collimating, and for finding the zenith or horizontal point of circles, used in the measurement of vertical angles. It consists of a collimating telescope T mounted in a vertical position upon an annular plate R, of cast-iron, floating upon the free surface of mercury, contained in an annular trough S, also of cast-iron. The annular plate is called the *float*. The telescope is mounted upon the float in a manner similar to the transit, except that the axis is nearer to the object end. One of the Ys may be elevated or depressed by an adjusting screw A, while the telescope is turned about its axis by another A', thus affording the means of giving the line joining the cross wires and the optical centre of the lens a vertical position. L is the lamp, and G the reflector, to catch its light and throw it upon the cross wires at the lower end of the tube.

Fig. 24.



2. — *The collimating process.*—Take the transit for instance. Level the axis carefully; turn the telescope in a vertical position; place the collimator below, and bring the image of the intersection of its cross wires, seen upon the bright ground G, accurately on the intersection of the middle wires in the transit, by means of the adjusting screws of the collimator; next turn the float in azimuth through 180°. If the emergent rays from the collimator be vertical, the image of the intersection of the collimator's wires will remain stationary, but if not, the image will move in the circumference of a circle; because, the plane of floatation

remaining the same, the emergent rays from the collimator will preserve their inclination to the horizon unchanged, thus causing the line through the optical centre of the transit's lens, and parallel to these rays, to describe a conical surface. The axis of this cone, which is a vertical line, is the position for the line of collimation. Supposing, then, the image to have changed its position during the semi-rotation of the float, renew the contact of the image and wires; one half by the adjusting screws of the collimator, and the other half by a motion of the transit and the adjusting screws of the diaphragm of its wires. This process being repeated once or twice, the adjustment is made.

3. — *The zenith or horizontal points.*—Direct the telescope of any circle to the collimator, and bring the image of the intersection of the cross wires in the collimator to the line of collimation; read the circle, and revolve the float through an azimuth of 180°; renew the contact of the image line of collimation by moving the circle, if necessary, and read again; denote the first reading by a, the second by a', and that of the zenith point by z, and we have

$$z=180°+\frac{a+a'}{2};$$

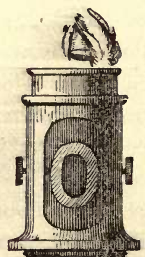
and denoting the reading of the horizontal point by h,

$$h=\frac{a+a'}{2}\pm 90°.$$

The Collimating Eye-piece.

4. — If now the swing-frame and its reflector be transferred from the collimating telescope to the eye-piece of the telescope of the instrument supposed to be vertical over a basin of mercury, this latter telescope becomes its own vertical collimator by reflection, on applying the lamp to the swing reflector. By perforating the swing reflector, and applying the eye behind it, two sets of wires will be seen in the solar focus of the telescope, and the collimating process consists in making the wires of one of these sets coincident with those of the other, by the joint motion of the telescope and its reticle. The little swing reflector, with a single microscope as an eye-piece, just behind its perforation, to magnify the wires and their images, constitutes the *collimating eye-piece.* This beautiful little instrument, which has done so much to facilitate the process of collimating and the measurement of zenith or nadir distances, is due to Professor Bohnenberger of Tübingen.

Fig. 25.

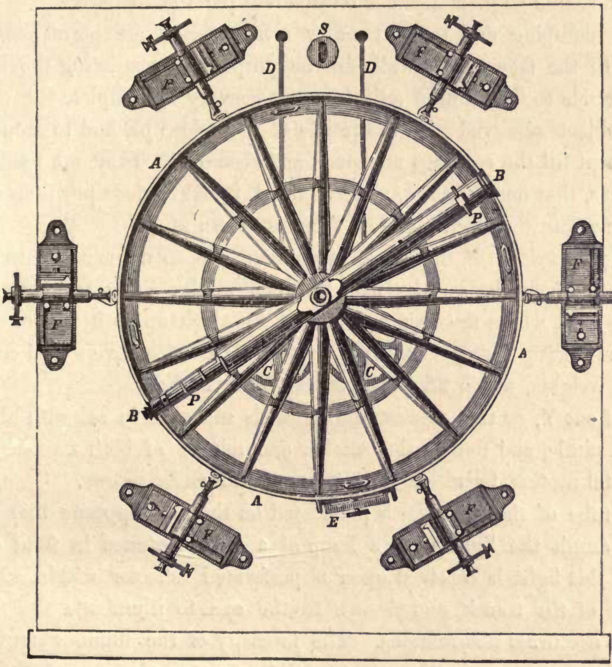


The Mural Circle.

1. — By means of the transit and a time-keeper, distances are measured on the equinoctial in time; and by an easy reduction this time is converted into arc. The object of the *Mural Circle* is to measure distances on the meridian.

This instrument consists of a metallic circle *A A*, varying in diameter from four to eight feet, strongly framed together or cast in one entire piece, and a telescope, of considerable optical power, having a focal length about equal to the diameter of the circle. The circle is firmly attached

Fig. 26.



to the larger end of a hollow conical-shaped axis at right angles to its plane, which axis is mounted on Ys, placed in an opening through a heavy wall, whose front face is in the plane of the meridian. The graduation is usually, though not always, upon the outer rim, and the readings are made by a pointer and six or more reading microscopes *F*, mounted upon the face of the wall, at equal distances from each other, around the circle. The telescope is mounted upon the front face of the circle, so as

to move parallel to the plane of the latter by means of a second axis, which turns freely and concentrically within that of the circle. The axis of the telescope is also conical, and is kept in place and proper contact with that of the circle, by means of a strong nut, which receives a screw cut upon its smaller end, the head of the nut bringing up against the end of the circle's axis. By turning this screw in the direction of its thread, the two axes are brought as closely in contact as may be found desirable.

Permanently connected with each end of the telescope is a clamping arrangement, for the purpose of seizing the rim of the circle, and when these are in bearing, the telescope can only move with the circle, and when loose, it may move independently, thus affording the means of measuring the same angular distance on different parts of the circle.

Five clamping and tangent screw arrangements are permanently attached to the face of the wall, for the purpose of restricting the motion of the circle to the minute adjustments necessary to complete the contact of the objects observed with the reticle of the telescope, and to secure the instrument till the readings are made and recorded. They are made thus numerous, that one may always be at hand, in the various positions of the observer about the circle; one of them is shown at *E*.

The proportions of the whole instrument are so adjusted as to throw its centre of gravity on the axis just behind the circle, and between it and the wall, where the axis is received by a stirrup with friction-rollers *C C*, the stirrup being connected by rods *D D* with levers and counterpoising weights, which take the bearing from the Ys.

The front Y, or that nearest the circle, is movable in azimuth about a vertical pintle, and that at the smaller end admits of both a vertical and horizontal motion, by means of two sets of antagonist screws.

The tube of the telescope is perforated on the side opposite that of the axis to admit the light from a lamp at a short distance in front of the circle; this light is received upon a perforated reflector within, after the manner of the transit, and thrown to the eye to illuminate the field of view in nocturnal observations. The intensity of the illumination is regulated by square perforations in two sliding plates, placed over the aperture in the tube, and so connected with rack and pinion work as to move in opposite directions, on turning a large milled-headed screw near the eye-glass; one of the diagonals of each square being placed in the direction of the motion of the plates, the figure of the opening will be unchanged, while its size may be varied at pleasure.

At *P* and *P* are two small tubes, permanently fixed to that of the telescope, and at right angles to its length. They are cut away on one

side at the middle, and each is closed at one end by a small disk of mother-of-pearl, movable about an axis perpendicular to its plane, and concentric with the tube. Between the disk and middle of the tube is a convex lens, which admits of a motion in the direction of the tube, and by which an image of a small eccentric perforation in the disk is formed about the middle of the cut, and of course on one side of the axis. A motion of the pearl causes this image to describe the circumference of a circle, of which the centre is on the axis of the tube. In the opposite end of the tube is a small microscope to view this image. The image is technically called the *ghost*, being a visible but unsubstantial representation of the perforation.

A small metallic style projects from the face of the wall at S, from the end of which may be suspended a plumb-line of fine silver wire, with its *bob* immersed in a vessel of water or other liquid at the bottom of the wall. The style is so arranged by an adjusting screw as to bring the plumb-line to intersect the axes of the small tubes in the cuts, or to throw it clear of the instrument, at pleasure.

In the tail end of the telescope, and at the solar focus of the object-glass, is a reticle, of which the axial wires are parallel to the axis of the circle. An additional wire is driven by a micrometer screw in the direction, perpendicular to the axial wires, while it is also kept constantly parallel to them.

The telescope has a collimating eye-piece, which is used for the same purpose and in the same manner as in the transit.

Adjustments.

2. — The adjustments are, first, to make the line of collimation perpendicular to the axis, and, second, to make the axis perpendicular to the meridian. The plane of the circle and tube of the telescope are placed at right angles to the axis by the manufacturer; the face of the wall is built as nearly in the meridian as possible by the aid of meridian marks; and the Ys are so placed as to bring the axis, when mounted, nearly perpendicular to the face, so that the adjustments are approximately made when the instrument is put up. To complete them, begin with

3. — *The line of collimation.*—Turn the circle till the telescope is vertical, suspend the plumb-line and bring it by its adjusting screw to coincide with the upper ghost as seen through the microscope: examine the position of the lower ghost; if it be not on the line, turn the pearl about its axis till it is: clear the line from the instrument, and invert the telescope by revolving the circle through 180°; bring the line to the

upper ghost as before, and again examine the lower ghost; if it be on the line, the axis of the circle is horizontal, but if not, bring it to the line, one-half by the vertical adjusting screws of the circle's axis and half by a revolution of the pearl. When by repeating this process once or twice the axis is made horizontal, put on the collimating eye-piece, and directing the telescope to the trough of mercury at the foot of the pier, and immediately below, move the diaphragm of the cross wires till the wire, which is perpendicular to the axis, coincides with its image—the line of collimation will be in a vertical plane, and of course perpendicular to the axis, which is horizontal.

Should the telescope have no collimating eye-piece, recourse may be had to the vertical collimator, which is to be used exactly as in the transit.

Since reflection takes place in a plane normal to the reflecting surface, the axis may be made horizontal by observing the same star directly, and by reflection from the free surface of mercury. If the time of the star's appearing on the line of collimation in both views be the same, the two positions of the line of collimation will lie in the same vertical plane, and being equally inclined to the horizon, the axis with which they make a constant angle must be horizontal.

4. — *Axis perpendicular to the meridian.*—This adjustment may be made by the method pointed out for the same adjustment in the transit; and when not perfected, the amount of error may be found by the process explained for that instrument.

Polar and horizontal points.—On the circumference of the circle is a scale of equal parts, each part having an angular value of five minutes. Every twelfth division is numbered, the numbers varying from 1 to 360° inclusive; these indicate the degrees of the scale; and to facilitate the reading, the intermediate divisions are also numbered, but in smaller characters.

If the reading be known when the line of collimation is either horizontal or directed to the pole of the heavens, and the reading be taken when directed upon the centre of any body as it passes the meridian, the difference of the readings will in the first case be the observed meridian altitude of the body, and in the second its observed polar distance.

5. — *The horizontal point.*—This is found by means of the collimating eye-piece, or vertical collimator, by the process indicated at page 268, or as follows, viz.: having carefully ascertained the value of a revolution of the micrometer in the eye-piece of the telescope, and the reading of its divided head when the movable wire is coincident with that parallel to he axis, set the telescope nearly in the position at which a star would

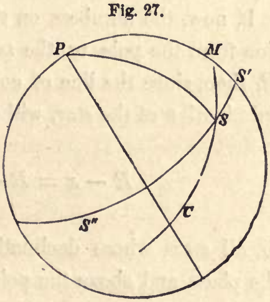
appear by reflection on the stationary wire; clamp the circle and record the reading of the index and microscopes; when the star is at a convenient distance from the meridian wire, bisect it by the movable wire without moving the circle, and note the time accurately. Unclamp the circle, and bring the star by direct view accurately on the stationary wire, by turning the whole circle about its axis; again note the time, and record the reading by the index and microscopes. Denote by R the first reading, by D the second, and by m the angular value of the distance between the fixed and movable wire, as indicated by the micrometer; then, if the star had been observed accurately on the meridian, would the reading of the horizontal point be

$$\frac{R \pm m + D}{2},$$

since the star must appear as far below the horizon by reflection as it actually is above it. But as the star cannot be taken at the same instant in both positions of the instrument, the readings R and D, taken as above indicated, must be reduced to what they would have been if taken on the meridian.

6. — This correction will now be explained.

Fig. 27.



Let $S' S S''$ be the small diurnal circle of the star; $P M S'$ an arc of the meridian; S the position of the star when observed on the intersection of the axial and one of the side normal wires; $M S C$ the arc of a great circle, of which the axial wire is a portion. The point M will be that to which the line of collimation is actually directed, and S' is that in which the star will reach the meridian; the arc $M S'$ is, therefore, the *reduction to the meridian.*

Make $P = MPS =$ hour angle of star;

$d = PS =$ polar distance of star;

$y = PM =$ polar distance of line of collimation;

$x = MS' =$ reduction to meridian.

Then in the triangle MPS, right-angled at M,

$$\cos P = \tan y \,.\, \cot d = \frac{\sin y}{\cos y} \cdot \frac{\cos d}{\sin d};$$

and subtracting this from

$$1 = 1,$$

we have, after reducing, and replacing $1-\cos P$ by $2\sin^2\frac{1}{2}P$,

$$2\sin^2\tfrac{1}{2}P=\frac{\sin(d-y)}{\cos y\,.\,\sin d}.$$

The observation being made very near the meridian, P and $d-y$ will be very small, and hence

$$\begin{aligned}2\sin^2\tfrac{1}{2}P &= 2\,.\,(\tfrac{1}{2}P\,.\,\sin 1'')^2=\tfrac{1}{2}P^2\,.\,\sin^2 1'';\\ \sin(d-y) &= \sin x = x\,.\,\sin 1'';\\ \sin d\,.\,\cos y &= \tfrac{1}{2}\sin 2\,d,\ \text{very nearly.}\end{aligned}$$

which in the above equation give, after reduction,

$$x=\tfrac{1}{4}\sin 2\,d\,.\,P^2\,.\,\sin 1'',$$

in which P is expressed in seconds of arc. To express it in time, make $P=15\,P_1$, and we shall finally have

$$x=\frac{225}{4}\,.\,\sin 2\,d\,.\,P_1^2\,.\,\sin 1'',$$

P_1 denoting the number of seconds of time in the hour angle of the star.

If, now, the numbers on the circle be supposed to increase in the direction from the pole to the zenith, and the observed reading be denoted by R, then, since the line of collimation is nearer the pole than the place of culmination of the star, will the true reading be

$$R-x=R-\frac{225}{4}\sin 2\,d\,.\,P_1^2\sin 1'' \quad . \quad . \quad . \quad . \quad (21)$$

for all stars whose declinations are of the same name as the latitude of the place, and above the pole, and

$$R+x=R+\frac{225}{4}\sin 2\,d\,.\,P_1^2\sin 1''\,. \quad . \quad . \quad . \quad . \quad (22)$$

for all stars below the pole, or whose declinations are not of the same name as the latitude.

7.— *The interval* P is obtained from the indications of a time-keeper. This usually runs too fast or too slow. To get the true from the indicated interval, suppose the time-keeper to gain or lose a seconds during one revolution of the earth upon its axis. Denote by A the number of sidereal seconds in the time of this revolution, and by t the true interval sought; then will

$$A \pm a : A :: P : t,$$

$$t = \frac{A}{A \pm a} . P = \frac{1}{1 \pm \frac{a}{A}} . P;$$

in which P is the indicated interval.

Developing the coefficient of P, and limiting the series to the first power of $\frac{a}{A}$, because a is usually a small number of seconds, we have

$$t = \left(1 \mp \frac{a}{A}\right) P;$$

or replacing A by its value 86.400,

$$t = (1 \mp .000012\, a)\, P = \alpha . P,$$

in which

$$\alpha = 1 \mp .000012\, a.$$

Substituting $\alpha\, P$ for P_1 in equations (21) and (22), and making

$$i = \alpha^2 = 1 \mp .000023\, a,$$

there will result for the true reading

$$R \mp \frac{225}{4} . i . P^2 . \sin 2\, d \sin 1'' \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

8. — Denote by D and R the readings of the circle by the direct and reflected views; by x and x' the corresponding reductions to the meridian; by m the small difference observed between the angle of incidence and reflection, and by H the reading of the horizonta. point; then will

$$H = \frac{D - x \pm m + R + x'}{2} = \frac{D + R}{2} \pm \frac{m}{2} - \frac{x - x'}{2},$$

and

$$H = \frac{D + R}{2} \pm \frac{m}{2} - \frac{225}{4} . i . \sin 1'' . \sin 2\, d\, (P^2 - P'^2) \quad . \ (24)$$

9. — *Value, in arc, of units on the screw-head connected with the movable wire.*—Run the movable wire to one edge of the field of view, say the upper, and bring it by a motion of the circle upon some well-defined and distant object; read the circle and micrometer; run the wire to the opposite or lower edge of the field, and by a motion of the circle bring the wire to same object again; read the circle and micrometer as before, and divide the difference of the circle readings, reduced to seconds, by the difference of the micrometer readings, expressed in units of the screw-head; the quotient will be the value sought. Or,

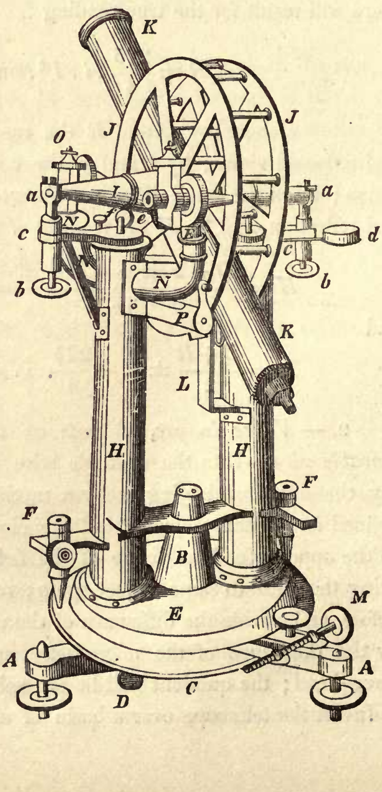
Invert the telescope over a basin of mercury, by moving the circle, and

bring the image of the movable wire, supposed at one edge of the field, to coincide with the wire itself; read the circle and micrometer: move the wire to the opposite edge, and turn the circle till the wire and its image again coincide, and read as before; divide the difference of the circle readings, reduced to seconds, by the difference of the micrometer readings expressed in units of its screw-head; the quotient will be the value sought.

Altitude and Azimuth Instrument.

1. — This instrument, as its name indicates, is employed in the measurement of vertical and horizontal angles. It has two graduated circles and a telescope. The planes of the circles are at right angles to each other; one called the azimuth circle, being connected with a tripod, by which it is levelled and kept in a horizontal position; while the other, called the altitude circle, is mounted upon a horizontal axis, with which the telescope is also united, after the manner of the transit.

Fig. 28.



To the centre of the tripod *A A* is fixed a vertical axis, of a length equal to about the radius of the circle; it is concealed from view by an exterior cone *B*. On the lower part of the axis, and in close contact with the tripod, is centred the azimuth circle *C*, which admits of a horizontal circular motion of about three degrees, for the purpose of bringing its zero exactly in the meridian; this is effected by a slow moving-screw, the milled head of which is shown at *D*. This motion should, however, be omitted in instruments destined for exact work, as the bringing the zero into the meridian is not requisite, either in astronomy or surveying: it is, in fact, purchasing a convenience too dearly, by introducing a source of error

not always trivial. Above the azimuth circle, and concentric with it, is placed a strong circular plate *E*, which carries the whole of the upper works, and also a pointer, to show the degree and nearest five minutes to be read off on the azimuth circle; the remaining minutes and seconds being obtained by means of the two reading microscopes *F*. This plate, by means of the cone *B*, rests on the axis, and moves concentrically with it. The conical pillars *H* support the horizontal or transit axis *I*, which, being longer than the distance between the centres of the pillars, the projecting pieces *c*, fixed to their top, carry out the Ys *a*, to the proper distance, for the reception of the pivots of the axis; the Ys are capable of being raised or lowered in their sockets by means of the milled-headed screws *b*, for a purpose hereafter to be explained. The axis, with its load, is prevented from pressing too heavily on its bearings, by two friction-rollers, on which it rests; one of these rollers is shown at *e*. A spiral spring, fixed in the body of each pillar, presses the rollers upward, with a force nearly a counterpoise to the superincumbent weight; the rollers on receiving the axis yield to the pressure, and allow the pivots to find their proper bearings in the Ys, relieving them, however, from a great portion of the weight.

The telescope *K* is connected with the horizontal axis, as before remarked, in a manner similar to that of the transit instrument. Upon the axis, as a centre, and in contact with the telescope on either side, is fixed the double circle *J*. The circles are united by small brass bars; by this circle the vertical angles are measured, and the graduations are cut on a narrow ring of silver, inlaid on one of the sides, which is usually termed the *face* of the instrument: a distinction essential in making observations. The clamp for fixing, and the tangent-screw for giving a slow motion to the vertical circle, are placed beneath it, between the pillars *H*, and attached to them, as shown at *L*. A similar contrivance for the azimuth circle is represented at *M*. The reading microscopes for the vertical circle are supported by two arms bent upward near their extremities, and attached to one of the pillars. The projecting arms are shown at *N* and the microscopes above at *O*, the latter admitting of a slight motion by means of antagonistic adjusting screws independently of the supporting arms.

A reticle consisting of five equidistant axial and as many equidistant normal wires, is in the principal focus of the object-glass. The illumination of the wires at night is by a lamp, supported near the top of one of the pillars at *d*, opposite the end of one of the pivots of the axis, which, being perforated, admits the light to the centre of the telescope tube,

where, falling on a diagonal reflector, it is reflected to the eye, and illumines the field of view.

The vertical circle is usually divided into four quadrants, each numbered 1°, 2°, 3°, &c., up to 90°, and following one another in the same order of succession; consequently, in one position of the instrument altitudes are read off, and with the face of the instrument reversed, zenith distances; and an observation is not to be considered complete till the object has been observed in both positions. The sum of the two readings will always be 90°, if there be no error in the adjustments, in the circle itself, or in the observations.

It is necessary that the microscopes *O* and the centre of the circle should occupy the line of its horizontal diameter; to effect which, an up-and-down motion, by means of the screws *b*, is given to the Ys. A spirit-level *P* is suspended from the arms which carry the microscopes: this shows when the vertical axis is set perpendicular to the horizon. A scale, usually showing seconds, is placed along the glass tube of the level, which exhibits the amount, *if any*, of the inclination of the vertical axis. This should be noticed repeatedly whilst making a series of observations, to ascertain if any change has taken place in the position of the instrument after its adjustments have been completed. One of the points of suspension of the level is movable, up or down, by means of the screw *f*, fo the purpose of adjusting the bubble. A striding-level, similar to the one employed for the transit instrument, and used for a like purpose, rests upon the pivots of the axis. It must be carefully passed between the radial bars of the vertical circle to set it up in its place, and must be removed as soon as the operation of levelling the horizontal axis is performed. The whole instrument stands upon three foot-screws, placed at the extremities of the three branches which form the tripod, and brass cups are placed under the spherical ends of the foot-screws. A stone pedestal, set perfectly steady, is the best support for this as well as the portable transit instrument.

Adjustments.

2. — These have for their object to make, 1st, *the azimuthal axis perpendicular* to the horizon; 2d, to make the axis of the vertical circle horizontal; 3d, to place the vertical circle at such a height that its microscopes shall point to the opposite extremities of a horizontal diameter; 4th, to make the line of collimation perpendicular to the axis of the altitude circle, and horizontal when the reading of the vertical circle is zero.

3. — *The vertical axis.*—Turn the instrument about, until the spirit-

level P is lengthwise in the direction of two of the foot-screws, when by their motion the spirit-bubble must be brought to occupy the middle of the glass tube, which will be shown by the divisions on the scale attached to the level. Having done this, turn the instrument half round in azimuth, and if the axis is truly vertical, the bubble will again settle in the middle of the tube; but if not, the amount of deviation will show double the quantity by which the axis deviates from the vertical in the direction of the level; this error must be corrected, one-half by means of the two foot-screws, and the other half by raising or lowering the spirit-level itself, which is done by the screw represented at *f*. The above process of reversion and levelling should be repeated, to ascertain if the adjustment has been correctly performed.

Next turn the instrument round in azimuth a quarter of a circle, so that the level P shall be at right angles to its former position; it will then be over the third foot-screw, which may be turned until the air-bubble is again central, if not already so, and this adjustment will be completed; if delicately performed, the air-bubble will steadily remain in the middle of the level during an entire revolution of the instrument in azimuth. These adjustments should be first performed approximately, for if the third foot-screw is much out of the level, it will be impossible to get the other two right. The vertical axis is now adjusted.

4. — *The axis of the vertical circle.*—This adjustment is performed exactly as in the transit, by means of the striding-level.

5. — *Height of the vertical circle.*—The last adjustment being made, bring the microscopes to their zeros, and turn the vertical circle slightly, the striding-level being still mounted, till some one of its divisions be brought to the cross wires of one of the microscopes. Examine the other microscope, and if its cross be not on or near the division of the circle, 180° distant from the first, depress or elevate the circle by the milled screws *b* till it is, keeping the axis horizontal by means of the level; this will give a sufficient approximation to bring the error of adjustment within the range of the adjusting screws which move the microscopes independently of their supporting arm. Recourse must now be had to these screws, by turning which in the direction indicated by the relative position of the circle division in question and the cross wires, the adjustment is perfected.

6. — *The line of collimation.*

As the vertical circle is not, like the mural, generally used as a differential instrument, but in the measurement of absolute altitudes or zenith distances, it is not only necessary that the line of collimation shall be per-

pendicular to the transit axis, but also that it shall be parallel to the radius of the graduated circle drawn to the zero of its scale.

Let x denote the angle made by the line of collimation with the plane normal to the transit axis, which angle is usually very small, and a the reading of the azimuth circle, when the telescope is pointed to some well-defined object in or near the horizon. If the line of collimation lie on the side of the normal plane, towards the zero of the circle, the true reading will be sensibly equal to

$$a - x,$$

if there be no other error of adjustment.

Now revolve the instrument in azimuth 180°, bring the telescope again on the object, and denote by a' the new reading; the true reading now will be

$$a' + x;$$

the difference of these true readings is obviously a semi-circumference, whence

$$a - a' - 2\,x = 180^\circ;$$

and

$$x = \frac{a - a' - 180^\circ}{2},$$

and the true reading in the second position becomes

$$a' + \frac{a - a' - 180^\circ}{2}.$$

Again, denote by y the small angle which the line of collimation makes with the plane passing through the axis of the vertical circle and that zero of this latter circle nearest the line of collimation, and suppose the line of collimation to lie above this plane when the telescope is directed to the same object, as before. Let b denote the apparent altitude, supposing the circle in the position to mark altitudes; the true altitude is sensibly equal to

$$b + y;$$

turn the instrument in azimuth 180°, and bring the telescope again on the object; the line of collimation will now be below the plane of the axis and zero, but the circle now indicates a zenith distance b', whence the true zenith distance is

$$b' + y;$$

adding these measures together, we have

$$b + b' + 2y = 90°$$

$$y = \frac{90° - (b + b')}{2},$$

and the true zenith distance becomes

$$b' + \frac{90° - (b + b')}{2}.$$

Whence to adjust the line of collimation we have this rule, viz.:

Direct the telescope to some well-defined and distant object, not far from the horizon, and bring its image to the intersection of the middle wires; record the reading of the azimuth and vertical circles; turn the instrument in azimuth 180°, bring the line of collimation again on the object, and record the new readings of the circles; subtract from the difference of the azimuthal readings 180°, divide by 2, and add (algebraically) the quotient to the last azimuthal reading for a new reading in azimuth. Add the two readings of the vertical circle together, subtract the sum from 90°, and add half the difference to the last reading for a new reading on the vertical circle. Set the circles to these new readings, clamp, and by the adjusting screws of the reticle bring the line of collimation to the object, and the adjustment is made; it should be verified, however, by repetition.

7. — To make the normal wires perpendicular to the transit axis, proceed as in the case of the transit instrument, viz.: Move the diaphragm about in its own plane, till the image of some object appears to run accurately along some one of the wires, say the middle one, while the telescope is turned about its axis.

8. — The altitude and azimuth instrument is regarded by many as the most universally useful of all astronomical instruments. It is portable and accurate. When used in the meridian, it may perform the work of the transit and mural circle, though with somewhat diminished accuracy. But its principal merit consists in the ease with which it may be moved in azimuth without impairing its measurement of altitudes and zenith distances.

9. — The *instrumental bearing* of an object is the angle indicated by the reading of the azimuth circle when the centre of the object is apparently on the line of collimation. From the instrumental, the true bearing, or true meridian, is found by a process to be explained hereafter.

10. — To find the altitude and instrumental bearing of an object at any instant, it is only necessary to make the object pass the line of collimation by turning both tangent-screws as it moves through the field of view, and to note the time of passage, and read the circles.

11.—The altitude and time, or the instrumental bearing and time, are the elements more commonly observed in the case of celestial objects.

12.—*To obtain the altitude and time.*—With the circles unclamped, direct the telescope, which it will be remembered inverts, so as to bring the image of the object in the lower or upper part of the field of view, as the body may be rising or setting; clamp the circles, and by the tangent screw of the *azimuth* motion, bring the image to the middle normal wire, and keep it there till it passes all the axial wires, carefully noting the time of its passing each, and also noting the indications of the level before it passes the first and after it passes the last one. Now read the vertical limb, unclamp, and, by an azimuthal motion, reverse the face of the vertical circle without unnecessary loss of time, and go through the same operation as before. Reduce the vertical readings to the same denomination of altitude or zenith distance, correct them by applying the level readings, and take half the sum for the altitude or zenith distance, as the case may be. Add the times together, and divide the sum by the number of recorded times for the corresponding time.

13.—*To find the instrumental bearing and time*, direct the telescope as before, and clamp; with the tangent-screw of the *vertical* motion, bring the image of the object to the middle axial wire, and keep it there till it passes all the normal wires, on each of which record the time. The reading of the azimuth circle will give the instrumental bearing, and a mean of all the times will give the corresponding time.

All of this supposes that the object's change in altitude and azimuth is uniform; and although this is not strictly true, it is nevertheless so nearly so for the short time its image is in the field of view, that the error will be inappreciable during the interval required for a single set of observations.

The Equatorial.

1.—The object of the equatorial or parallactic, as it is frequently called, is to support a telescope, generally of great size and optical power, in such manner as to give to the observer the means of directing it with ease to any part of the heavens, and to measure at once the apparent hour angle and polar distance of a heavenly body. In the principles of its construction, it is like the altitude and azimuth instrument, but differs from it in the position of its axes, which, instead of being vertical and horizontal, are, when in position, respectively perpendicular, and parallel to the plane of the equinoctial. The first is called the *polar*, the second the *declination axis.* It has two graduated circles, one securely attached to each

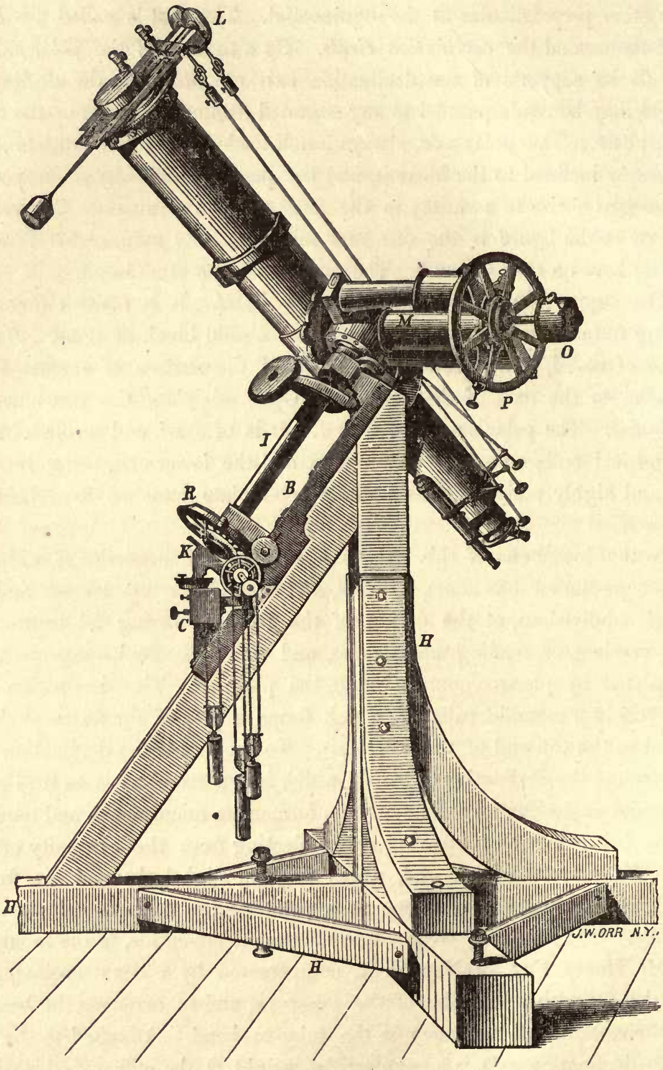
axis; the plane of one, viz., that attached to the polar axis, is parallel, and the other perpendicular to the equinoctial. The first is called the *hour*, and the second the *declination circle*. By a motion of the polar axis, to which the supports of the declination axis are attached, the declination circle may be made parallel to any assumed declination circle of the celestial sphere. The polar axis, always much loaded, is, in low latitudes, considerably inclined to the horizon, and the practical difficulty of supporting it has given rise to a variety in the form of the instrument. That represented in the figure is the one now most generally used, and it is introduced here on that account. The principle is the same in all.

The supporting-stand is shown at *H*, *H*, *H*. It is made either of a strong frame of wood-work, or is cut from a solid block of stone. *B* is a plate of metal, firmly secured to the stand, the surface of contact being parallel to the axis of the heavens. Upon this plate the instrument is mounted. The polar axis is seen at *I*. It is of steel, and revolves in two cylindrical collars near the extremities, and the lower end, being rounded off and highly polished, rests upon a steel plate attached to a bearing-piece *K*.

To the lower end of this axis is attached the hour-circle *R*, which is either graduated into hours, minutes, and seconds, or into degrees and the usual subdivisions, at the option of the person ordering the instrument. The verniers, or reading microscopes, and tangent-screw arrangement, are supported by pieces connected with the plate *B*. The declination axis revolves in a metallic tube *M*, which forms a part of the frame-work secured to the top end of the polar axis. To one end of the declination axis is attached the declination circle *P*, which is graduated so as to read polar distances or declinations—suppose the former, its micrometers and tangent-screw being mounted upon pieces projecting from the extremity of the tube *M*, and to the other end, which projects slightly beyond the frame-work, is attached the telescope at a point nearer the eye-end than the middle. The excess of weight towards the object-end is, in the mounting by Mr. Henry Fitz, of New York, compensated by a counterpoise cylindrical lever within the tube of the telescope, and so arranged in bearing as to counteract all tendency in the tube to bend. Attached to the end of the declination axis, is a counterpoise weight *O*, the office of which is to throw the centre of gravity of the entire movable part of the instrument in the polar axis near its upper end, where it is received by a pair of friction-rollers.

At *C* is a box containing a system of wheel-work, so connected with the polar axis as, by the aid of weights and a centrifugal governor, to give

Fig. 29.



it a uniform motion of rotation. The velocity of rotation is regulated by a vertical motion of the axis of the governor, whose balls in their retrocession and increasing velocity, force a pair of rubbing surfaces against the interior of an inverted conical box: the moment of the friction thence arising equilibrates that of a descending weight, and the motion becomes

uniform. By elevating the axis of the governor, the motion is accelerated; by depressing the axis, it is retarded, and thus the velocity of rotation may be made equal to that of the earth about its axis, in which case a star in the field of view will be kept there by the instrument itself, the effect being the same, abating refraction, as though the earth were at rest.

2. — With a divided object-glass for the telescope, to be explained presently, or with the position micrometer, the equatorial is mostly used as a differential instrument, and particularly when the observer is provided with a very full and accurate catalogue and map of the stars, which serve as points of reference. Whenever it is possible to bring a known object into the field of view with one that is not known, the place of the latter is found by measuring its bearing and distance from the known object.

3. — To measure directly the hour angle and polar distance of an object with the equatorial, requires the parts of the instrument to be in perfect adjustment among one another, and its polar axis to be parallel to the axis of the earth. For these adjustments and a full analysis of the equatorial.

Analysis of the Equatorial.

The *true* instrumental position of an object is that indicated by an instrument in perfect adjustment within itself. The *apparent* instrumental position is that actually indicated by an instrument whether in adjustment or not. When the several parts of an instrument are adjusted with respect to each other, these two positions are the same.

The instrumental *hour angle* of an object, is its angular distance from a vertical plane passing through the polar axis, estimated upon the hour circle.

Its instrumental *declination* is its angular distance from a plane perpendicular to the polar axis, estimated upon the declination circle; and its instrumental polar distance, its angular distance from the polar axis.

The line of collimation should be perpendicular to the declination axis, and the latter perpendicular to the polar axis. The index of the hour circle should stand at the zero of the scale when the line of collimation is parallel to the vertical plane of the polar axis, and, supposing the instrument to read polar distances, the index of the declination circle should be at the zero of its scale, when the line of collimation is parallel to the polar axis.

Supposing none of the conditions to be fulfilled, the apparent instrumental position of an object will differ from the true, and the first thing to be done is to find the latter from the former, when the error in each of the above particulars is known. To do this, we will premise that the equatorial may be regarded as an universal transit instrument, whose horizon is the equinoctial, and zenith the pole. The formulæ of reduction applicable to the transit will apply at once to the equatorial by making therein the symbol for the latitude 90°; in which case we shall have for the difference between the true and apparent instrumental hour angle in arc, the sum of the last three terms of Eq. (15), viz.,

$$\frac{c}{\cos\delta} + l \, . \, \frac{\cos(\lambda - \delta)}{\cos\delta} + z \, . \, \frac{\sin(\lambda - \delta)}{\cos\delta};$$

which reduces, by making $\lambda = 90°$, and replacing δ by $90° - \pi$, to

$$c \, . \, \text{cosec}\, \pi + l \, . \, \cot \pi + z;$$

in which c is the error in the line of collimation, l that of the declination axis, and z a constant correction to be applied to every reading of the hour circle arising from the improper position of its index, and therefore the index error of the hour circle, and π the instrumental polar distance of an object whose image is on the line of collimation.

Denoting by σ' the true, and by σ the apparent instrumental hour angle, and writing $\varDelta\sigma$ for z, we have

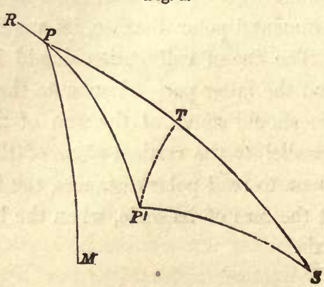
$$\sigma' = \sigma + \varDelta\sigma + l \, . \, \cot\pi + c \, . \, \text{cosec}\,\pi \quad . \quad . \quad . \quad . \quad (a)$$

Denoting by π' the true instrumental polar distance, and by $\varDelta\pi$ the index error, then will

$$\pi' = \pi + \varDelta\pi \, . \; . \; . \; . \; . \; . \; . \; . \; . \quad (b)$$

Let us next find from the hour angle and polar distance as given by the instrument, whose parts are in perfect adjustment among themselves, the *true hour angle* and *polar distance*, referred to the true meridian and pole of the celestial sphere. It is obvious that the instrumental and true co-ordinates would not differ, if the polar axis of the instrument were parallel to that of the heavens. Suppose this latter condition not fulfilled, and that the inclination of these axes is very small, as it always is, after putting the instrument in position

Fig. 1.



after the manner to be explained presently. Let PM be the arc of the meridian; P, the true pole of the heavens; P', the point in which the polar axis of the instrument produced pierces the celestial sphere; and S, the position of a star. Make

$$p = PS \quad = \text{the true polar distance;}$$
$$\pi' = P'S \quad = \text{the instrumental polar distance;}$$
$$s = MPS = \text{the true hour angle of the star.}$$

The difference between the true and instrumental hour angles will be sensibly equal to the difference of the angles which the true and instrumental declination circles of the object make with the plane of the celestial and instrumental axes, or $SPR - SP'R$; PR being $P'P$ produced. Make

$\lambda = PP'$;

$\varphi = MPP'$, positive when this angle is to the west of the meridian

$c = SPR$;

$d = PSP'$.

Then in the triangle SPP', we have, *Napier's Analogies*,

$$\tan \tfrac{1}{2}(P + P') = \cot \tfrac{1}{2} d \, . \, \frac{\cos \frac{1}{2}(p - \pi')}{\cos \frac{1}{2}(p + \pi')}.$$

But $P = 180° - c$, and replacing $\cot \frac{1}{2} d$ by its value in terms of sin and cos, we have

$$\tan [90° - \tfrac{1}{2}(c - P')] = \cot \tfrac{1}{2}(c - P') = \frac{\cos \frac{1}{2} d}{\sin \frac{1}{2} d} \, . \, \frac{\cos \frac{1}{2}(p - \pi')}{\cos \frac{1}{2}(p + \pi')};$$

taking the reciprocal, and observing that the angles $\frac{1}{2}(c - P')$, $\frac{1}{2} d$, and $\frac{1}{2}(p - \pi')$ are very small, and that $p = \pi'$, we have very nearly

$$c - P' = s - \sigma' = d \, . \cos \pi' \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

But from the same triangle we have

$$\sin d = d = \lambda \, . \, \frac{\sin P}{\sin \pi'};$$

and this in Eq. (c), observing that the angle $P = s - \varphi$, gives

$$s - \sigma' = \lambda \, . \sin (s - \varphi) \, . \cot \pi';$$

transposing and replacing σ' by its value in Eq. (a), we have, since s and σ only differ by a small quantity,

$$s = \sigma + \Delta\sigma + \lambda \sin (\sigma - \varphi) \cot \pi + c \, . \operatorname{cosec} \pi + l \cot \pi \quad . \quad . \quad (d)$$

With S as a pole, and radius SP', describe the arc $P'T$, then will

$$PS = p = \pi' + PT.$$

But within the limits supposed

$$PT = \lambda \,.\, \cos\,(s - \varphi) = \lambda \,.\, \cos\,(\sigma - \varphi)\,;$$

whence, replacing π' by its value in Eq. (b), we have

$$p = \pi + \varDelta\pi + \lambda \,.\, \cos\,(\sigma - \varphi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

Adjustments.

The adjustments of the equatorial are of two classes, viz.: those which relate to the parts among one another; and those which determine the position of the instrument in relation to the celestial sphere.

The rules for the first are suggested by equations (a) and (b), and are as follows:

Index Error of the Declination Circle.—Direct the line of collimation to any well-defined object in any part of the horizon; in reversed positions of the declination circle, the readings of this circle in Eq. (b) give

$$\pi' = \pi + \varDelta\pi;$$
$$\pi' = \pi_{,} - \varDelta\pi.$$

Taking the second from the first,

$$\varDelta\pi = -\frac{\pi - \pi_{,}}{2}.$$

Apply this with its proper sign to the last reading $\pi_{,}$, and the telescope still being upon the object, move the verniers or microscopes till they indicate this corrected reading.

Line of Collimation.—The preceding correction being applied, move the telescope till the declination circle marks a polar distance equal to 90°; then by a motion of the polar axis, bring the line of collimation upon some object directly in the instrumental east or west; read the hour circle; reverse the declination circle; bring the telescope upon the same object, and read again; and these readings, in Eq. (a), will give, since $\pi = 90°$,

$$\sigma' = \sigma + \varDelta\sigma + c,$$
$$\sigma' = \sigma_{,} - 12^{h} + \varDelta\sigma - c;$$

whence, subtracting the second from the first,

$$c = -\frac{\sigma - \sigma_{,} + 12^{h}}{2}.$$

Apply this to the last reading $\sigma_{,}$, and move the instrument about its polar

axis till the vernier indicates this reading; then by a motion of the adjusting screws which act upon the telescope, bring the line of collimation to bear upon the object.

Declination Axis.—Turn the line of collimation to an object directly in the instrumental north or south, to get the greatest declination. This will give to l its greatest effect. Read the hour circle as before in the direct and reversed position of the declination circle. Then, since by the last adjustment $c = 0$, we have

$$\sigma' = \sigma + \varDelta\sigma + l \,.\, \cot\pi,$$
$$\sigma' = \sigma_{,} - 12^{h} + \varDelta\sigma - l \cot\pi;$$

whence, by subtraction and reduction,

$$l = -\frac{\sigma - \sigma_{,} + 12^{h}}{2} \,.\, \tan\pi;$$

or, if the telescope be set to a polar distance equal to 45°,

$$l = -\frac{\sigma - \sigma_{,} + 12^{h}}{2}.$$

Set the hour circle to the last reading $\sigma_{,}$, corrected by the above value of l, and bring the line of collimation back to the object by the adjusting screws, which act upon the declination axis.

The Polar Axis parallel to the Axis of the Heavens.—About the time that some circumpolar star, the nearer the pole the better, comes to the meridian—say its upper passage—turn the declination circle till it reads the star's polar distance, increased by the refraction due to its altitude, and clamp the declination circle; then by a motion of the entire instrument in right ascension, and the screws which act upon the polar axis in the meridian, bring the star to the cross wires, and keep it there till the instant, as indicated by a time-piece, of its crossing the meridian. This will be sufficient for the first approximation.

Then observe some well-known star in quick succession very near the meridian, reversing the declination circle. The reading of the declination circle, corrected for refraction, will give, in Eq. (e), since $\sigma = 0$,

$$p = \pi + \varDelta\pi + \lambda \,.\, \cos\varphi,$$
$$p = \pi_{,} - \varDelta\pi + \lambda \,.\, \cos\varphi;$$

whence

$$\lambda \,.\, \cos\varphi = p - \frac{\pi + \pi_{,}}{2}.$$

The first member being the projection of the arc λ on the meridian, is the

arc by which the pole is too high or too low. The axis being moved through this distance by estimation, direct the telescope to the polar distance, corrected for refraction, of a second star soon to come to the meridian; when the star is in the field put the clock movement in motion, and as the star culminates, as indicated by a time-piece, bring the axial wire to the star by the adjusting screws of the polar axis which are in the meridian.

The polar distance of another star when six hours from the meridian being observed in quick succession, in the direct and reversed position of the declination circle, Eq. (e) gives, since in this case $\sigma = 90°$,

$$p = \pi + \varDelta\pi + \lambda \,.\, \sin\varphi,$$
$$p = \pi_{,} - \varDelta\pi + \lambda \sin\varphi;$$

whence

$$\lambda \,.\, \sin\varphi = p - \frac{\pi + \pi_{,}}{2}.$$

The first member is the projection of the arc λ, on the declination circle at right angles to the meridian, and is, therefore, the deviation of the pole of the instrument from this latter plane. This error being treated in a manner similar to the preceding, by means of adjusting screws which act at right angles to the meridian, the polar axis is brought to this latter plane, and the instrument will be so nearly in adjustment as to bring the errors within the limitations required to render equations (d) and (e) exact.

The approximation may be continued, if desirable, or the value of each error found by recourse to celestial objects properly selected, and these errors employed as elements of reduction.

To find c.—Observe an equatorial star about the time of its meridian passage, and again after reversing the declination circle; the readings of the hour circle in Eq (d) give

$$s = \sigma + \varDelta\sigma + c \operatorname{cosec}\pi,$$
$$s' = \sigma' - 12^{\text{h}} + \varDelta\sigma - c \operatorname{cosec}\pi;$$

whence

$$c = \frac{(s - \sigma) - (s' - \sigma') - 12^{\text{h}}}{2} \,.\, \sin\pi.$$

Denote by a the right ascension of the star, and by t and t' the sidereal times of observation, we have

$$s = t - a; \quad s' = t' - a;$$

these in the above equation give

$$c = \frac{(t - t') - (\sigma - \sigma') - 12^h}{2} \sin \pi \quad . \quad . \quad . \quad (f)$$

To find l.—Observe some star, near the pole, in quick succession reversing the declination circle; the readings of the hour circle in Eq. (d) give

$$s = \sigma + \varDelta\sigma + \lambda \sin(\sigma - \varphi) \cot \pi + c \,.\, \mathrm{cosec}\, \pi + l \cot \pi,$$
$$s' = \sigma' - 12^h + \varDelta\sigma + \lambda \sin(\sigma' \quad 12^h - \varphi) \cot \pi - c \,.\, \mathrm{cosec}\, \pi - l \cot \pi;$$

whence, since the third terms of the second members do not differ sensibly,

$$l \,.\, \cot \pi + c \,\mathrm{cosec}\, \pi = \frac{(s - s') - (\sigma - \sigma') - 12^h}{2};$$

eliminating s and s' by their values $t - a$, and $t' - a$, and reducing,

$$l = \frac{[\overline{t - t'} - \overline{\sigma - \sigma'} - 12^h] \,.\, \sin \pi - 2c}{2 \,.\, \cos \pi} \quad . \quad . \quad . \quad . \quad (g)$$

To find φ *and* λ.—Observe any well-known star, and again after reversing the declination circle. The readings of the circles in Equations (d) and (e) give

$$s = \sigma + \Delta\sigma + \lambda \,.\, \cot \pi \,.\, \sin(\sigma - \varphi) + n,$$
$$s' = \sigma' - 12^h + \Delta\sigma + \lambda \,.\, \cot \pi \sin(\sigma' - 12^h - \varphi) + n',$$
$$p = \pi + \Delta\pi + \lambda \,.\, \cos(\sigma - \varphi),$$
$$p' = \pi_{,} + \Delta\pi + \lambda \,.\, \cos(\sigma' - 12^h - \varphi);$$

in which

$$n = c \,.\, \mathrm{cosec}\, \pi + l \,.\, \cot \pi,$$
$$n' = c \,.\, \mathrm{cosec}\, \pi_{,} + l \,.\, \cot \pi_{,}.$$

Subtracting the first from the second, the third from the fourth, transposing, and making, after eliminating s' and s by their equals $t' - a$, $t - a$,

$$\Sigma = (t' - t) - (\sigma' - \sigma - 12^h) - (n' - n),$$
$$\Pi = (p' - p) - (\pi_{,} - \pi),$$

we obtain

$$\left.\begin{array}{l} \lambda \,.\, \cot \pi\, [\sin(\sigma' - 12^h - \varphi) - \sin(\sigma - \varphi)] = \Sigma \\ \lambda \,.\, [\cos(\sigma' - 12^h - \varphi) - \cos(\sigma - \varphi)] = \Pi \end{array}\right\} \quad . \quad (h)$$

but

$$\sin(\sigma' - 12^h - \phi) - \sin(\sigma - \phi) = 2 \sin \tfrac{1}{2}(\sigma' - \sigma - 12^h) \,.\, \cos\left(\frac{\sigma' + \sigma - 12^h}{2} - \phi\right),$$

$$\cos(\sigma' - 12^h - \phi) - \cos(\sigma - \phi) = 2 \sin \tfrac{1}{2}(\sigma' - \sigma - 12^h) \,.\, \sin\left(\frac{\sigma' + \sigma - 12^h}{2} - \phi\right).$$

substituting these above, and dividing the second equation by the first, we have, using p for π,

$$\tan\left(\frac{\sigma' + \sigma - 12^h}{2} - \varphi\right) = \frac{\Pi}{\Sigma} \cot p \quad . \quad . \quad . \quad . \quad (i)$$

whence

$$\varphi = \frac{\sigma' + \sigma - 12^h}{2} - \tan^{-1} . \frac{\Pi}{\Sigma} \cot p \quad . \quad . \quad . \quad . \quad (k)$$

and from equations (h) we have

$$\lambda = \frac{\frac{1}{2}\Pi}{\sin\frac{1}{2}(\sigma' - \sigma - 12^h) . \sin\left(\frac{\sigma' + \sigma - 12^h}{2} - \phi\right)} = \frac{\frac{1}{2}\Sigma}{\sin\frac{1}{2}(\sigma' - \sigma - 12^h) . \cos\left(\frac{\sigma' + \sigma - 12^h}{2} - \phi\right)\cot p} \quad (l)$$

To find $\Delta\,\sigma$.—Observe a star before its culmination in the hour angle $360° - \sigma$, and at an interval after its culmination in the hour angle σ', such, that $360° - \sigma$ and σ' shall be equal, or very nearly so, without reversing the declination circle ; Eq. (d) will then give

$$24^h - s = 24^h - \sigma + \Delta\,\sigma + \lambda \cot \pi \sin\,(360° - \overline{\sigma + \varphi}) - n,$$
$$s' = \sigma' + \Delta\,\sigma + \lambda \cot \pi . \sin\,(\sigma' - \varphi) + n.$$

Adding and reducing,

$$s' - s = \sigma' - \sigma + 2\,\Delta\,\sigma + \lambda \cot \pi . [\sin\,(\sigma' - \varphi) - \sin\,(\sigma + \varphi)] ;$$

writing $\sin\,(\sigma - \varphi)$ for $\sin\,(\sigma' - \varphi)$, to which it is sensibly equal, we have, after developing the last term, reducing, and replacing s and s' by their equals $t - a$ and $t' - a$,

$$\Delta\,\sigma = \frac{(t' - t \; - (\sigma' - \sigma)}{2} + \lambda . \cot \pi . (\sin \varphi . \cos \sigma) \quad . \quad . \quad (m)$$

For a star in or near the equator, we may take $\cot \pi = 0$; or for a star whose hour angle is 90°, in which case $\cos \sigma = 0$, the above value for index error becomes

$$\Delta\,\sigma = \frac{(t' - t) - (\sigma' - \sigma)}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (m')$$

To find $\Delta\,\pi$.—Observe the same star twice in quick succession, and in reversed positions of the declination circle ; the readings of the declination circle, in Eq. (e), give

$$p = \pi + \Delta\,\pi + \lambda \cos\,(\sigma - \varphi),$$
$$p = \pi' - \Delta\,\pi + \lambda \cos\,(\sigma - \varphi) ;$$

whence by subtraction,

$$\Delta\,\pi = \frac{\pi' - \pi}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (n)$$

Heliometer.

1. — The image formed by a lens of a point on the surface of an object, is on a line drawn through the optical centre of the lens and the point. If the point be stationary and the lens in motion, along a line perpendicular to this line, the image will also be in motion, and in the same direction.

Every fragment cut from a lens by a section parallel to its axis, forms an image just as large and as perfect as does the entire lens, the only difference being in the intensity of its illumination, which will be less in proportion as the surface of the fragment is less than that of the entire lens.

If, then, the lens be divided by a plane through its optical axis, and the two halves moved in opposite directions, and perpendicular to this axis, an image of an object formed by the entire lens will be duplicated, and the individuals of the pair will be equally bright. Two half lenses, so mounted as to be moved parallel to the dividing plane, called *the plane of section,* and at right angles to the optical axis, by means of micrometer screws, constitute the *Heliometer.* Such an arrangement forms the object-glass of the telescope at L, in Fig. 29. The screws are furnished with large circular heads, which are carefully graduated after the manner of those of the position micrometer, and are turned by the aid of a rod, reaching to the eye-end of the telescope. The entire frame-work, which supports the slides of the semi-lenses, admits of a rotary motion about the axis of the telescope's tube, and is put in motion by a second rod, also passing to the eye-end. By this last arrangement the plane of section may be made to pass through any two objects, whose images are simultaneously in the field of view.

2. — The value in arc of the linear distance through which the images of the same object are made to separate, by turning the micrometer screw-head through each unit of its scale, is found by a process in all respects similar to that explained in Appendix No. I., for the position micrometer.

3. — Directing the telescope to the sun, duplicating its image, and turning the micrometer screws till the images are tangent, the reading multiplied by the angular value of the head unit will gi e the apparent diameter of the sun. Hence the name of the instrument.

4. — But it is obvious that the apparent dimensions of any other body may be measured in the same way. Also the angular distance subtended by the line, joining two objects, whose images may be brought into the field of view together. For this purpose, turn the whole field lens till the plane of section pass through the objects, duplicate the image of both, and turn the micrometer screws till one of the images of the one be brought to coincide with an image of the other; the reading, treated as before, will give the angular distance sought.

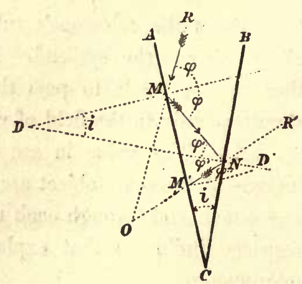
The Sextant.

1. — This is employed in the measurement of the angular distance between two objects. It is one of the most generally useful instruments that has yet been devised, furnishing, as it does, data for the solution of a variety of astronomical problems of the greatest practical utility both on land and at sea. It is especially useful at sea, where the unstable position of the mariner excludes the use of almost all other instruments.

It depends upon this catoptrical principle, viz.: When a ray of light is reflected by two plane reflectors in a plane normal to both, the ray is deviated or bent from its original direction through an angle equal to twice the angle made by the reflectors.

Let $A\,C$ and $C\,B$ represent the section of two plane reflectors perpendicular to their line of intersection C. $R\,M$, $M\,N$, and $N\,O$ the course of a ray reflected first at the point M, and next at the point N; then will the angle $R\,O\,R' = 2\,A\,C\,B$. For, draw the normals $M\,D$, $M'\,D'$ to the reflector $A\,C$, and $D\,D'$ to the reflector $C\,B$, and denote by φ and φ'' the angles of incidence on the reflector $A\,C$; by φ' that on the reflector $C\,B$, and by i the inclination of the reflectors. Then, since by the principle of optics the angle of incidence is equal to that of reflection, we have from the triangle $M\,D\,N$,

Fig. 80.



$$\varphi - \varphi' = i;$$

and from the triangle $M'\,D'\,N$,

$$\varphi' - \varphi'' = i;$$

adding these, we have

$$\varphi - \varphi'' = 2\,i;$$

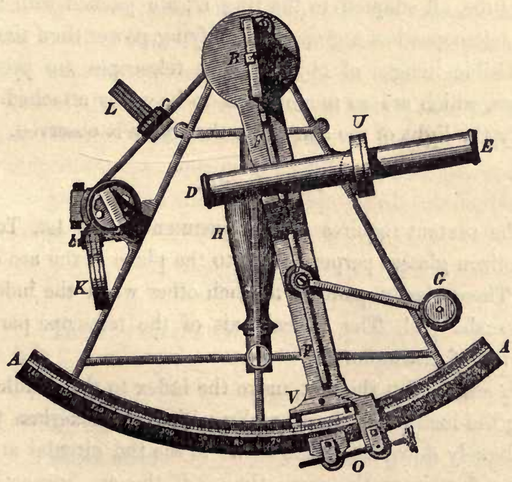
and because $M\,D$ and $M'\,D'$ are parallel, the first member is the inclination of the first incident to the second reflected ray.

If then the reflector CB were transparent at the point N, the waves of light from an object at R', would be transmitted through it and coincide in direction with those from R reflected at M and N; and to an eye situated at O, the objects R and R' would apparently coincide. Two reflectors so mounted as to give the means of reading their inclination to each other, when this coincidence takes place, would give the angular distance $R\,O\,R'$ of the objects by simple inspection; and, with appliances to facilitate the operations of the observer, constitute a reflecting instrument, which, according as its arc of measurement is extended to an entire circumference or limited to an arc of 90°, 60°, or 45°, is called a *reflecting circle, quadrant, sextant,* or *octant.* The sextant is the more common of the instruments with limited arcs now in use.

2.— The annexed figure represents a sextant. It consists of the two plane-glass reflectors C and B seen edgewise; a graduated arc $A\,A$, of which the plane is perpendicular to those of the reflectors; an index-arm F, vernier V, clamp and tangent screw O; a telescope $E\,D$, of which the line of collimation is parallel to the plane of the arc of measurement; colored glasses L and K to qualify the light received into the telescope, and a triangular system of frame-work uniting strength with lightness, to support all the parts and render them available. The handle of the instrument is represented at H.

The arc of measurement is divided into half-degree spaces, which are numbered as whole degrees, and these divisions are subdivided to any de-

Fig. 31.



sirable extent consistent with facility of reading. The reflector B, called the *index-glass*, is covered with an amalgam of tin on the face towards the eye-end of the telescope, and turns with the index-arm about an axis in its own plane, and through the centre of the arc of measurement, being perpendicular to the plane of the latter. The reflector C, called the horizon-glass, is, abating the limited range of the adjusting screws, securely fixed with its plane also at right angles to that of the arc of measurement. Only one-half of this glass is covered, and that half lies nearest the frame of the instrument, the covered face being turned from the telescope. The line separating the covered from the uncovered part of this glass is parallel to the plane of the graduated arc, and at a distance therefrom about equal to that of the line of collimation, being sometimes a little greater and sometimes a little less in consequence of a change in the position of the telescope, to make the supply of light it receives through the uncovered, equal to that which enters it after reflection from the coated part of the horizon-glass. The position of the telescope is altered by means of a screw and milled nut connected with its supporting ring U. By turning the nut the telescope is thrust from or drawn towards the face of the sextant. This device is called the *up-and-down piece*. There are usually six or seven colored glasses of different shades, which are so mounted that they can be turned about an axis c or b parallel to the face of the sextant, and be interposed or not at pleasure.

To facilitate the reading, a small microscope G is attached to a swing movable about an axis a, connected with the index-arm. Two telescopes and a plane tube, all adapted to the ring U, are packed with the sextant. One of these telescopes has a greater magnifying power than the other, and inverts the visible images of objects. The telescopes are provided with colored glasses, which are so mounted as to be easily attached to the eye-end to qualify the light of the sun when that body is observed.

Adjustments.

3. — The sextant requires three adjustments, viz.: 1st. To make the index and horizon glasses perpendicular to the plane of the arc of measurement. 2d. These glasses parallel to each other when the index is at the zero of the scale. 3d. The optical axis of the telescope parallel to the plane of the arc of measurement.

4. — To accomplish the first, move the index to the middle of the arc, then holding the instrument horizontally with the index-glass towards the eye, look obliquely down this glass so as to see the circular arc by direct view and by reflection at the same time. If the arc appear broken, the

position of the glass must be altered till it appear continuous, by means of small screws that attach the frame of the glass to the instrument.

The horizon-glass is known to be perpendicular to the plane of the instrument when, by a sweep of the index, the reflected image of an object and the image seen directly, pass accurately over each other; and any error is rectified by means of an adjusting screw, provided for the purpose, at the lower part of the frame of the glass.

5. — The second adjustment is effected by placing the index or zero point of the vernier to the zero of the limb; then directing the instrument to some distant object (the smaller the better), if it appear double, the horizon-glass must, after easing the screws that attach it to the instrument, if there be no adjusting screw for the purpose, be turned around a line in its own plane and perpendicular to that of the instrument, till the object appear single; the screws being tightened, the perpendicular position of the glass must again be examined. The adjustment may, however, be rendered unnecessary by correcting an observation by the *index error.* The effect of this error on an angle measured by the instrument is exactly equal to the error itself: therefore, in modern instruments, there are seldom any means applied for its correction, it being considered preferable to determine its amount previous to observing, or immediately after, and apply it with its proper sign to each observation. The amount of the index error may be found in the following manner: clamp the index at about 30 minutes to the left of zero, and looking towards the sun, the two images will appear either nearly in contact or overlapping each other; then perfect the contact, by moving the tangent-screw, and call the minutes and seconds denoted by the vernier, the reading on the arc. Next place the index about the same quantity to the right of zero, or on the arc of excess, and make the contact of the two images perfect as before, and call the minutes and seconds on the arc of excess the reading off the arc; half the difference of these numbers is the index error; additive when the reading on the arc of excess is greater than that on the limb, and subtractive when the contrary is the case.

Example.

		$'$	$''$
Reading on the arc	. . .	31	56
" off the arc	. . .	31	22
Difference	. . . . .	0	34
Index error	. . . $= -$	0	17

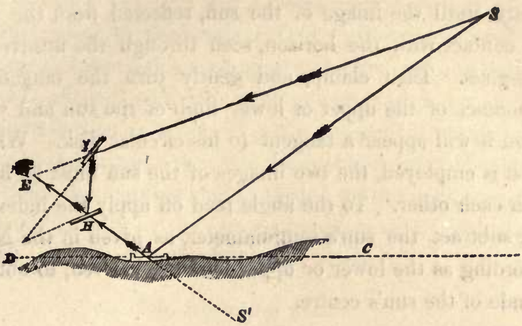
In this case the reading on the arc being greater than that on the arc of excess, the index error, = 17 seconds, must be subtracted from all observations taken with the instrument, until it be found, by a similar process, that the index error has altered. One observation on each side of zero is seldom considered enough to give the index error with sufficient exactness for particular purposes: it is usual to take several measures each way; "and half the difference of their means will give a result more to be depended on than one deduced from a single observation only on each side of zero." A proof of the correctness of observations for index error is obtained by adding the above numbers together, and taking one-fourth of their sum, which should be equal to the sun's semidiameter, as given in the Nautical Almanac. When the sun's altitude is low, not exceeding 20° or 30°, his horizontal instead of his perpendicular diameter should be measured (if the observer intends to compare with the Nautical Almanac, otherwise there is no necessity); because the refraction at such an altitude affects the lower border (or limb) more than the upper, so as to make his perpendicular diameter appear less than his horizontal one, which is that given in the Nautical Almanac: in this case the sextant must be held horizontally.

6. — The third adjustment is made by the aid of two parallel wires placed in the common focus of the telescope for the purpose of directing the observer to the centre of the field of view, in which an observation should always be made; these wires are parallel to the plane of the instrument, and divide the field of view into three nearly equal parts. The sun and moon are made tangent to each other, when their angular distance is 90° or more, at one of the wires; the position of the sextant is then altered so as to bring these bodies to the second wire; if the contact continue, the line of collimation is parallel to the plane of the instrument; if not, the position of the telescope must be altered by means of two adjusting screws connected with the up-and-down piece.

Artificial Horizon.

7. — To measure directly the altitude of any celestial object with the sextant, it would be necessary that the object and horizon should be distinctly visible; but this is not always the case in consequence of the irregularity of the ground which conceals the horizon from view. The observer

Fig. 32.



is therefore obliged to have recourse to an artificial horizon, which consists usually of the reflecting surface of some liquid, as mercury contained in a small vessel A, which will arrange its upper surface parallel to the natural horizon $D A C$. A ray of light $S A$, from a star at S, being incident on the mercury at A, will be reflected in the direction $A E$, making the angle $S A C = C A S'$ ($A S'$ being $E A$ produced), and the star will ap pear to an eye at E as far below the horizon as it actually is above it Now with a sextant whose index and horizon glasses are represented at I and H, the angle $S E S'$ may be measured; but $S E S' = S A S' - A S E$, and because $A E$ is exceedingly small as compared with $A S$, the angle $A S E$ may be neglected, and $S E S'$ will equal $S A S'$, or double the altitude of the object: hence one-half the reading of the instrument will give the apparent altitude. At sea, the observer has the natural or sea horizon as a point of departure, and the altitude may be measured directly.

8. — Having now gone through the principle and construction of the sextant, it remains to give some instructions as to the manner of using it.

Fig. 33.

It is evident that the plane of the instrument must be held in the plane of the two objects, the angular distance of which is required. The sextant must be held in the right hand, and as loosely as is consistent with its safety, for in grasping it too firmly the hand is apt to be rendered unsteady.

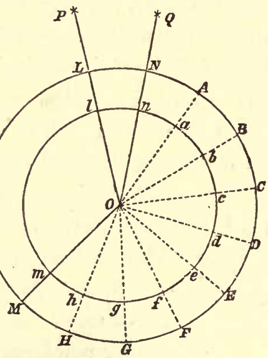
When the altitude of an object, the sun for instance, is to be observed, the observer, having the sea-horizon before him, must turn down one or more of the dark glasses or shades, according to the brilliancy of the object; and directing the telescope to that part of the horizon immediately beneath the sun, and

holding the instrument vertically, he must with the left hand slide the index forward, until the image of the sun, reflected from the index-glass, appears in contact with the horizon, seen through the unsilvered part of the horizon-glass. Then clamp, and gently turn the tangent-screw, to make the contact of the upper or lower limb of the sun and the horizon perfect, when it will appear a tangent to his circular disk. When an artificial horizon is employed, the two images of the sun must be brought into contact with each other. To the angle read off apply the index error, and then add or subtract the sun's semidiameter, as given in the Nautical Almanac, according as the lower or upper limb is observed, to obtain the apparent altitude of the sun's centre.

The Principle of Repetition.

1. — By this principle, the invention of Borda, the error of graduation in any instrument may be diminished, and, practically speaking, annihilated. Let $P\,Q$ be two objects which we may suppose fixed, for purposes of mere explanation, and let $O\,L$ be a telescope movable on O, the common axis of two circles, $A\,M\,L$ and $a\,b\,c$, of which the former $A\,M\,L$ is fixed in the plane of the objects, and carries the graduations, and the latter is freely movable on the axis. The telescope is attached permanently to the latter circle, and moves with it. An arm $O\,a\,A$ carries the index or vernier, which reads off the graduated limb of the fixed circle. This arm is provided with two clamps, by which it can be temporarily connected with either circle, and detached at pleasure. Suppose, now, the telescope directed to P. Clamp the index-arm $O\,A$ to the *inner* circle, and unclamp it from the outer, and read off. Then carry the telescope round to the other object Q. In so doing, the inner circle, and the index-arm which is clamped to it, will also be carried round, over an arc $A\,B$, on the graduated limb of the outer, equal to the angle $P\,O\,Q$. Now clamp the index to the outer circle, and unclamp the inner, and read off: the difference of readings will of course measure the angle $P\,O\,Q$; but the result will be liable to two sources of error—that of *graduation* and that of observation, both of which it is our object to get rid of. To this end transfer the telescope back to P, *without* unclamping the arm from the outer circle; *then*, having made the bisection of P,

Fig. 84.



clamp the arm to b, and unclamp it from B, and again transfer the telescope to Q, by which the arm will now be carried with it to C, over a second arc $B\,C$, equal to the angle $P\,O\,Q$. Now again read off; then will the difference between this reading and the *original* one measure *twice* the angle $P\,O\,Q$, affected with *both* errors of observation, but only with *the same error of graduation as before.* Let this process be repeated as often as we please (suppose ten times); then will the final arc $A\,B\,C\,M$ read off on the circle be ten times the required angle, affected by the joint errors of all the ten observations, but only by the same constant error of graduation, which depends on the initial and final readings off alone.

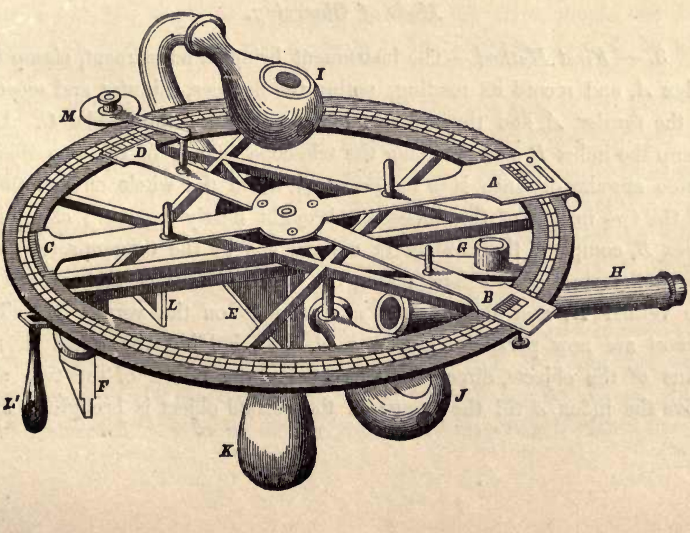
The Reflecting Circle.

1. — The use of this instrument is, in general, the same as that of the sextant; but when it unites, as it often does, to the catoptrical principle of this latter instrument, the principle of repetition, it becomes, in the hands of a skilful observer, one of the most refined and elegant of the portable implements in the service of astronomy.

This form of the instrument is represented in the annexed figure.

The arc of measurement, which is extended to the entire circumference, is divided into 720 equal parts, and, for the reason explained in the account of the sextant, these parts are numbered as whole degrees, the subdivisions being continued to any desirable degree of minuteness.

The circle is mounted upon two concentric axes, which may move independently of each other, and also of the circle. Upon one end of the



central axis is mounted a reflector E, similar to the index-glass of the sextant, and upon the other an arm $A\ C$, in the position of a diameter of the circle. Upon the corresponding ends of the other axis are mounted a system of frame-work and a second arm $B\,D$. This frame-work supports a second reflector F, similar to the horizon-glass of the sextant, a telescope H, colored glasses L and L', and the handles I, J, K for holding in different positions. The reflectors are perpendicular to the plane of the circle. Each of the arms $A\ C$ and $B\,D$ has a vernier at both ends, and at one end a vernier, clamp, and tangent-screw, so that the reflectors may be clamped in any position consistent with their being perpendicular to the plane of the circle, and for each position there will be two arc readings, differing by 180°.

At G is seen the barrel for the up-and-down piece, of which the milled head is concealed beneath the end B of the arm $B\,D$.

At M are seen the microscope and its reflector for reading, mounted upon a pin projecting from the vernier arm.

The circle is usually accompanied by a stand, to which it may be attached, when great steadiness is required, by means of screw holes in the handles; one of these holes is seen in the handle I.

Adjustments.

2. — The adjustments are the same as those of the sextant, and performed in the same manner, with the exception of the index error, which, in this instrument, is always eliminated by the manner of observing.

Mode of Observing.

3. — *First Method.*—The instrument being in adjustment, clamp the index A, and record its reading, noting the degrees, minutes, and seconds on the vernier A, and the minutes and seconds on the vernier C. Unclamp the index B, and directing the telescope to one of the two objects whose angular distance is to be measured, move the whole circle around till the two images of this object are brought nearly together; clamp the index B, complete the contact or coincidence by the tangent-screw, and record the reading as before, noting the degrees, minutes, and seconds on the vernier B, and the minutes and seconds on the vernier D. The glasses are now parallel. Unclamp A, and, holding the circle in the plane of the objects, direct the telescope to the fainter of the two, and move the index A till the image of the second object is brought nearly

in contact with that of the first; clamp, and complete the contact by the tangent-screw: read the verniers A and C as before. The difference of the A readings will give the angle as measured by the sextant, and this angle should always be noted as a check. Next, unclamp B, and keeping the telescope upon the same object, move the whole circle till the two images of this object are again nearly in contact; clamp, and finish the contact by the tangent-screw. The glasses are again parallel, and the index B has passed over an arc equal to the angular distance of the two objects. Unclamp A, and move it in the same direction as before till the two objects again appear nearly in contact; clamp, and complete the contact with the tangent-screw; the index of A will thus have passed over an arc equal to twice the angular distance of the objects. Now unclamp B, and turn the whole instrument as before till the two images of the same object again appear; clamp, and complete the contact, and the index of B will also have passed over an arc equal to twice the angular distance of the objects. This process being repeated as often as may be deemed desirable, finally read the verniers as before. Take a mean of the minutes and seconds of the first reading of A and C, as also of B and D; these with the degrees of A and B will give the true readings of the instrument at the beginning of the operation; do the same for the last reading, or that at the close of the repetitions. Take the difference between the last and first readings of the instrument for each set of verniers; add these differences together, and divide the sum by the number of times that A and B have been moved after the first contact of the images of the same object: the quotient will be the angle sought.

A comparison of this angle with that given by the difference of the second and first readings of A, will indicate the error, should one have been committed, either in the readings or in taking account of the number of repetitions.

Second Method.—Clamp A, and record the readings of A and C as before; unclamp B; direct the telescope to the fainter of the two objects, and turn the circle till the second object appear nearly in contact with the first; clamp B; complete the contact by the tangent-screw, and record the reading of B and D. Now, invert the instrument by revolving it through an angle of 180° about the line of collimation of the telescope; unclamp A, and move this index till the objects again appear nearly in contact; clamp, and complete the contact by the tangent-screw; the difference of the second and first readings of A will be double the angular distance of the objects, the half of which will be the check. Bring the instrument back to its former position by revolving it about the line

of collimation; unclamp *B*, and turn the circle till the images again appear; clamp, and complete the contact by the tangent-screw; the arc passed over by *B* will also be double that of the objects. This process being repeated as often as the observer pleases, finally read the instrument on both sets of verniers; take the first reading of *A* and *C* from the last; do the same for *B* and *D*; add these differences together, and divide the sum by twice the number of times that *A* and *B* have been moved since the first contact.

4.—The process of repeating is much facilitated by the following device. A brazen arc is attached to the frame-work of the instrument so as to be concentric with the arc of measurement, and just below it, and moves with the telescope and horizon-glass. It is out of view in the position of the instrument represented in the figure. To this arc are fitted two small sliders, that adhere to it by friction, wherever placed. Firmly attached to the tangent-screw end of the arm *AC* are two small pieces of metal, called checks, lying in the direction of radii, and just long enough to cross the brazen arc, and to slide over its surface, after the manner that the index moves over the arc of measurement, so that if one of the sliders be interposed, the motion of the index will be arrested.

5.—In the first method of observing, after the two images of the same object are made to coincide, place one of the sliders against the check on the side from which the index *A* must be moved to bring the other object in the field of view; after the contact of the two objects is perfected, by moving the index *A*, place the other slider in contact with the other check on the opposite side. Now, the circle being in the plane of the objects, a little consideration will make it manifest, that to restore the contact of the images of the same, and afterwards of the two objects, it will only be necessary to bring the checks in contact with their respective slides by alternately moving the circle and index *A*. The brazen arc is sometimes graduated and numbered in opposite directions, commencing from the positions of the checks, corresponding to the parallel position of the reflectors; this furnishes an additional check upon the angle measured, and facilitates the management of the sliders. The use of the sliders in the second method of observing is, from what has been said, too obvious to need explanation.

6.—This Appendix contains, it is believed, an account of all that is essential in the theory, construction, and use of the principal instruments employed in astronomical measurements. To describe all that are in use, would expand the work to dimensions inconsistent with its object, viz.: to give to students in the threshold, as it were, of Astro-

nomy, a preparation for future progress in the subject. The German Meridian Circle combines the Mural and Transit, as does also the English Transit Circle. One of the most useful instruments to which the student can give his attention, is the Zenith Telescope, alluded to on page 199 of the text.

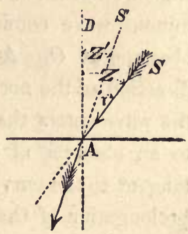
APPENDIX III.

ATMOSPHERIC REFRACTION.

When light passes from one medium to another it is refracted according to the law expressed by the equation, *Optics,* § 15,

$$\sin z = m \sin z' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Fig. 2.



in which z is the angle which the normal to the incident wave makes with the normal to the deviating surface, and z' the angle which the normal to the deviated wave makes with the same.

Denote the angle of deviation SAS' by r, then will

$$z' = z - r,$$

which substituted in Eq. (1), we have, after developing,

$$\sin z = m (\sin z \cos r - \cos z \sin r);$$

and because r is always small when m differs little from unity, which is the case in the passage of light through the different strata of the atmosphere, we may write

$$\cos r = 1, \text{ and } \sin r = r;$$

and dividing both members of the above equation by $\cos z$, we have

$$r = \frac{m-1}{m} . \tan z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and regarding z as constant, r will vary with m, and hence

$$dr = \frac{dm}{m^2} \tan z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now, if we regard the atmosphere as composed of indefinitely thin and concentric strata of increasing density from the top to the bottom, dr will be the deviation of the ray in passing from one stratum to another whose

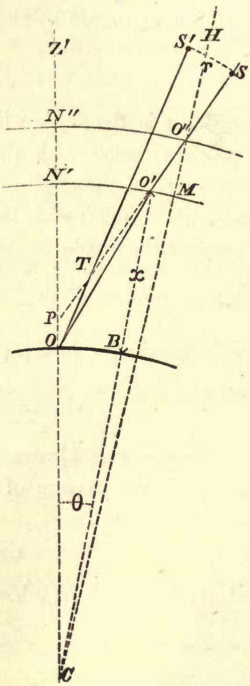
indexes of refraction differ by dm. But in the same kind of medium, this difference is found by careful experiment to be directly proportional to the difference of densities; hence

$$\frac{dm}{m^2} = \lambda \, dD,$$

in which λ is a constant and D the density of the atmosphere at the place of any one stratum whose index is m. Whence, Eq. (3),

$$dr = \lambda \, . \, dD \, . \tan z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Fig. 8.



Let OB be the arc of the earth's surface in a vertical plane through a heavenly body S; $O'N'$ and $O''N''$, two concentric strata of atmosphere in the same plane; $SO''O'O$, the curve which is normal to the front of a luminous wave coming from the body to the observer at O. As an object is seen in the direction of the normal to the wave from it as the wave enters the eye, the body will appear to an observer at O to be at S', on the line tangent to the curve at O; and if SP be the prolongation of the straight portion of the ray before it enters the atmosphere, the angle STS' will be the total deviation. This angle STS' is called *the refraction*. Denote the radius of the earth CO by unity; the height of the stratum $N'O'$ above the surface by x; the angle OCO' by θ. Then taking O' and O'' contiguous, we have $O'CO'' = d\theta$, $MO'' = dx$; and the angle of incidence $HO''S$, on the stratum of which $O''N''$ is the upper limit, being denoted by z, we have $MO' = dx \tan z$, and

$$d\theta = \frac{dx \tan z}{1 + x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

And denoting the apparent zenith distance $Z'OS'$ by Z, we have

$$r = TPZ' - Z = \theta + z - Z;$$

and by differentiating

$$dr = d\theta + dz;$$

or substituting the value of $d\theta$, in Eq. (5),

$$dr = \frac{dx \,.\, \tan z}{1+x} + dz;$$

whence, Eq. (4),

$$\lambda dD \,.\, \tan z = \frac{dx \,.\, \tan z}{1+x} + dz;$$

or

$$\frac{dz}{\tan z} = \lambda\, d\, D - \frac{dx}{1+x};$$

by integration,

$$\log \sin z = \lambda D - \log (1+x) + C;$$

and making $x = 0$, in which case $D = D_{,}$ and $z = Z$, we have

$$\log \sin Z = \lambda D_{,} + C;$$

and by subtraction,

$$\log \frac{\sin z}{\sin Z} = -\lambda\,(D_{,} - D) - \log (1+x),$$

or

$$\log \frac{\sin z}{\sin Z} = \log e^{-\lambda\,(D_{,}-D)} - \log (1+x);$$

whence

$$\sin z = \frac{\sin Z \,.\, e^{-\lambda\,(D_{,}-D)}}{1+x} \quad . \; . \; . \; . \; . \; . \quad (6)$$

But, Eq. (4),

$$dr = \lambda d D \tan z = \frac{\lambda \sin z \,.\, dD}{\sqrt{1-\sin^2 z}};$$

and substituting the value of sin z above,

$$dr = \frac{\lambda \,.\, \sin Z \,.\, e^{-\lambda\,(D_{,}-D)} \,.\, dD}{\sqrt{(1+x)^2 - \sin^2 Z \,.\, e^{-2\lambda\,(D_{,}-D)}}} \quad . \; . \; . \; . \quad (7)$$

If the law which connects the varying density D with the height x be given, one of these variables may be eliminated and the integration performed. But in a practical point of view this is not necessary; for λ is known to be a very small fraction, as is also the greatest value of x, the latter not exceeding 0.01931, being the height of the first stratum of air that has sensible action upon light, divided by the radius of the earth, or 77 miles divided by 4000 miles. Developing the factors $e^{-\lambda\,(D_{,}-D)}$ and $e^{-2\lambda\,(D-D)}$, neglecting the second and higher powers of λ and x, and also the term of which $\lambda \sin^2 Z$ is a factor, which may be done without sensible error when Z does not exceed 80°, we find

$$dr = \frac{\lambda \,.\, \sin Z dD}{\sqrt{1 + 2x - \sin^2 Z}} = \frac{\lambda \sin Z \,.\, dD}{\sqrt{\cos^2 Z + 2x}};$$

or

$$dr = \frac{\lambda \,.\, \tan Z \,.\, dD}{\sqrt{1 + 2x \sec^2 Z}} = \lambda \tan Z \,.\, (1 - x \sec^2 Z)\, dD;$$

whence

$$r = \lambda \tan Z \int (dD - \sec^2 Z x dD);$$

and performing the integration, that of the last term by parts,

$$r = \lambda \tan Z \left[D - \sec^2 Z \left(Dx - \int D \,.\, dx\right)\right];$$

but if h denote the height of the mercurial column at any stratum of air above the observer, $D_{\prime\prime}$ the density of the mercury, and g the force of gravity regarded as constant, then will

$$g \,.\int D dx = g D_{\prime\prime} h;$$

and

$$r = \lambda \tan Z\, [D - \sec^2 Z\, (Dx - D_{\prime\prime} h) + C];$$

and from the limit $x = 0$, where $D = D'$ and $h = h_{\prime}$, to the limit $x =$ height of the entire atmosphere, where $D = 0$, $r = 0$, and $h = 0$, we find

$$r = \lambda \tan Z \,.\, D' \left(1 - h \,.\, \frac{D_{\prime\prime}}{D'} \sec^2 Z\right).$$

Taking the density of Mercury as unity, we have the mean value of $D' = \frac{1}{10485}$.

The mean value of h is found from the proportion,

$$\overset{\text{miles}}{4000} : \overset{\text{inches}}{29.6} :: 1 : h;$$

which will give for the coefficient of $\sec^2 Z$,

$$h \,.\, \frac{D_{\prime\prime}}{D'} = 0.0012517.$$

Also, if $D_{\prime}$ be the density of air when the thermometer is 50, and the barometer 30 inches; and we take $\alpha = 0.00208$, and $\beta = 0.0001001$, the coefficients of expansion for air and mercury respectively, then, *Analytical Mechanics*, § 245,

$$D' = D_{\prime} \,.\, \frac{h}{30} \,.\, \frac{1 + (50 - t) \,.\, \beta}{1 + (t - 50) \,.\, \alpha}.$$

in which t denotes the actual temperature of the air and mercury supposed the same, and h the height of the barometer. Hence

$$r = \lambda D_{\prime} \cdot \frac{h}{30} \cdot \frac{1 + (50 - t)\beta}{1 + (t - 50)\alpha} \cdot \tan Z \, (1 - 0.0012517 \sec^2 Z) \; . \; . \; (8)$$

Had the second power of x been retained in Eq. (7), then would

$$r = \lambda D_{\prime} \cdot \frac{h}{30} \cdot \frac{1+(50-t)\beta}{1+(t-50)\alpha} \cdot \tan Z \left(1 - 0.0012517 \sec^2 Z + 0.00000139 \frac{2+\sin^2 Z}{\cos^4 Z}\right) (8)'$$

the last term of which, within the limits supposed, is insignificant.

Make

$$u = \frac{h}{30} \cdot \frac{1 + (50 - t)\beta}{1 + (t - 50)\alpha} \cdot \tan Z \, . \, (1 - 0.0012517 \sec^2 Z) \; . \; . \; (9)$$

and we have

$$r = \lambda D_{\prime} u \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (10)$$

Denote by z and z' the greatest and least observed zenith distances of a circumpolar star, r and r' the corresponding refractions, and c the zenith distance of the pole; then will

$$c = \frac{z + r + z' + r'}{2}.$$

In like manner, if $z_{\prime}$ and $z_{\prime}'$ be the greatest and least zenith distances of another circumpolar star, $r_{\prime}$ and $r_{\prime}'$ the corresponding refractions,

$$c = \frac{z_{\prime} + r_{\prime} + z_{\prime}' + r_{\prime}'}{2}.$$

Equating these values, replacing the refractions by the values given in Eq. (10), we find

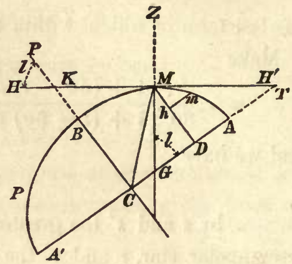
$$\lambda D_{\prime} = \frac{z_{\prime} + z_{\prime}' - (z - z')}{u + u' - (u_{\prime} - u_{\prime}')}.$$

The indications of the barometer and thermometer being substituted in Eq. (9), give u, u', $u_{\prime}$, and $u_{\prime}'$, and therefore the value of $\lambda D_{\prime}$. Numerous and careful observations make $\lambda D_{\prime} = 57''.82$, which substituted in equations (8) and (8)′, give the refraction for every observed zenith distance, temperature of the air, and height of the barometer.

APPENDIX IV.

SHAPE AND DIMENSIONS OF THE EARTH.

Fig. 4.



Let AMP, A' represent a meridional section of the terrestrial ellipsoid, M the place of the spectator, B the north pole of the earth, C its centre, Z the zenith, HMH' a parallel to the rational horizon and tangent to the meridian section at M, $A'A$ the intersection of the equator by the meridian plane.

Make

$l =$ the angle $MGA = PKH =$ latitude of M;
$A = CA$, the equatorial radius;
$B = CB$, the polar radius.

Then, referring the curve to the centre and axis, its equation is

$$A^2y^2 + B^2x^2 = A^2B^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

the equation of the tangent line HH',

$$A^2yy' + B^2xx' = A^2B^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

and the equation of the normal at M,

$$A^2y'(x - x') - B^2x'(y - y') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

in which x' and y' are the co-ordinates of M.

Denote the angle MTC by T, then from Eq. (b) we have

$$\tan T = \frac{B^2x'}{A^2y'};$$

but $T = 90° - l$, whence

$$B^2x' \tan l = A^2y' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

Also, denoting the eccentricity by e, we have

$$e^2 = \frac{A^2 - B^2}{A^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

Substituting $x'y'$ for xy in Eq. (a), combining the resulting equation with Eq. (d), and eliminating B by means of Eq. (e), we find

$$\left.\begin{aligned} x' &= \frac{A\cos l}{\sqrt{1-e^2\sin^2 l}} \\ y' &= \frac{A(1-e^2)\,.\,\sin l}{\sqrt{1-e^2\sin^2 l}} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (f)$$

Differentiating the first, regarding x and l as variable, we have

$$dx' = -A\,.\,\frac{(1-e^2)\sin l\,dl}{(1-e^2\sin^2 l)^{\frac{3}{2}}} \quad . \; . \; . \; . \; . \; . \; . \quad (g)$$

but, designating by s the linear dimension of any portion of the arc of the curve, we have for the projection of the element ds on the axis of x,

$$ds\,.\cos T = ds\,.\sin l;$$

and since x is a decreasing function of the latitude,

$$-dx' = ds\,.\sin l;$$

which substituted in Eq. (g) gives

$$ds = A\,.\,\frac{(1-e^2)\,dl}{(1-e^2\sin^2 l)^{\frac{3}{2}}} \quad . \; . \; . \; . \; . \quad (h)$$

For any other latitude l', we have

$$ds' = A\,.\,\frac{(1-e^2)\,.\,dl'}{(1-e^2\sin^2 l')^{\frac{3}{2}}} \quad . \; . \; . \; . \; . \; . \; . \quad (h)'$$

dividing the first by the second, making

$$dl = dl' = 1°,$$

and solving with respect to e^2, we find

$$e^2 = \tfrac{2}{3}\,.\,\frac{ds - ds'}{ds\sin^2 l - ds'\sin^2 l'} \quad . \; . \; . \; . \; . \quad (i)$$

From Eq. (h) we have

$$A = \frac{ds}{1-e^2}\,(1-e^2\sin^2 l)^{\frac{3}{2}} \quad . \; . \; . \; . \; . \; . \quad (j)$$

and from the well-known property of the ellipse,

$$B = A\sqrt{1-e^2} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (k)$$

Making $ds = c$, $ds' = c'$, $l = l_m$, $l' = l'_m$, we have equations (10) and (11) of the text.

Denoting by R the radius of curvature at any point of the meridian, we have

$$R = \pm \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx d^2 y};$$

finding the values of dx, dy, and $d^2 y$ from Eqs. (f), and substituting above, there will result

$$R = A \cdot \frac{1 - e^2}{(1 - e^2 \sin^2 l)^{\frac{3}{2}}}.$$

Then

$$2\pi R : 360° :: \beta : 1°;$$

whence

$$\beta = \frac{2\pi}{360} \cdot R = \frac{2\pi}{360} \cdot A \cdot \frac{1 - e^2}{(1 - e^2 \sin^2 l)^{\frac{3}{2}}} \quad . \quad . \quad . \qquad (l)$$

in which β denotes the linear dimension of one degree of latitude.

Denoting by ρ the radius of the earth in any latitude l, we obtain by squaring and adding Eqs. (f),

$$\rho = A \cdot \sqrt{1 - \frac{e^2 (1 - e^2) \sin^2 l}{1 - e^2 \cdot \sin^2 l}} \quad . \quad . \quad . \quad . \quad . \qquad (m)$$

Every section of the terrestrial spheroid through the centre is an ellipse of which the semi-transverse and semi-conjugate axes are respectively A and ρ, l being the latitude of the extremity of the conjugate axis. Denoting by $e_{\prime}$ the eccentricity of the elliptical section, we have

$$e_{\prime}^2 = \frac{A^2 - \rho^2}{A^2} = \frac{e^2 (1 - e^2) \sin^2 l}{1 - e^2 \sin^2 l};$$

this value of $e_{\prime}^2$ substituted in Eq. (l) after making therein $l = 90°$, and denoting by $\beta_{\prime}$ the length of a degree on the section perpendicular to the meridian in the latitude l,

$$\beta_{\prime} = \frac{2\pi}{360} \cdot A \cdot \sqrt{\frac{1 - e^2 \sin^2 l}{1 - e^2 (2 - e^2) \sin^2 l}} \quad . \quad . \quad . \quad . \qquad (n)$$

The value of the radius of the parallel of latitude is given by that of x', Eqs. (f); and denoting by α the linear length of a degree of longitude on this parallel, we have

$$\alpha = \frac{2\pi}{360} \cdot x' = \frac{2\pi}{360} \cdot A \quad \frac{\cos l}{\sqrt{1 - e^2 \sin^2 l}} \quad . \quad . \quad . \quad . \qquad (o)$$

Dividing both members of Eq. (a) by A^2B^2, making $A = 1$ and $B = \gamma$, that equation becomes

$$\frac{y^2}{\gamma^2} + x^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (p)$$

Differentiating, we find

$$-\frac{dx}{dy} = \frac{1}{\gamma^2} \cdot \frac{y}{x};$$

but the angle at M in the evanescent triangle $m\,M\,h$ is equal to the angle at $G = l'$ in the triangle $M\,G\,D$; and denoting in future the central latitude $M\,C\,D$ by l, we have

$$-\frac{dx}{dy} = \tan l',$$

$$\frac{y}{x} = \tan l;$$

whence

$$\tan l = \gamma^2 \tan l' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (q)$$

Making $A = 1$, $B = \gamma$, and eliminating e^2 from Eq. (m) by the relation $e^2 = 1 - \gamma^2$, we have

$$\rho = \frac{1}{\sqrt{1 + \frac{1 - \gamma^2}{\gamma^2} \cdot \sin^2 l}} \quad . \quad . \quad . \quad . \quad (r)$$

APPENDIX V.

EARTH'S ORBIT.

The sun's attraction for the earth varies inversely as the square of the distance. The earth describes, therefore, an ellipse about the sun, having the latter body in one of its foci.

By Eq. (266), *Analytical Mechanics*, we have

$$\frac{d\alpha}{dt} = \frac{2c}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

in which α denotes the angle which the radius vector of the earth makes with any assumed axis, r the radius vector, c the area described by the latter in a unit of time, and t the time.

Also, Eq. (277), *Analytical Mechanics*,

$$a(1-e^2)=\frac{4c^2}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

in which a is the semi-transverse axis of the earth's orbit, e its eccentricity, and k the intensity of the sun's attraction on the unit of mass of the earth at the unit's distance.

The polar equation of the ellipse is

$$r=\frac{a(1-e^2)}{1+e\cos V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

in which V is the true anomaly, estimated from the perihelion.

Eliminating r and c from Eq. (a) by means of Eqs. (b) and (c), we have

$$\frac{\sqrt{k}}{a^{\frac{3}{2}}}\,.\,dt=(1-e^2)^{\frac{3}{2}}\,.\,(1+e\cos V)^{-2}\,.\,d\alpha \quad . \quad . \quad . \quad (f)$$

developing the factors of the second members by the binomial formula, and neglecting all the terms involving the powers of e higher than the second, we have

$$\frac{\sqrt{k}}{a^{\frac{3}{2}}}\,.\,dt=(1-\tfrac{3}{2}e^2)(1-2e\cos V+3e^2\cos^2 V-\&c.)\,d\alpha$$

and because

$$\cos^2 V=\tfrac{1}{2}+\tfrac{1}{2}\cos 2V,$$

$$d\alpha=dV;$$

and, § 201, *Analytical Mechanics*,

$$\frac{\sqrt{k}}{a^{\frac{3}{2}}}=\frac{2\pi}{T}=m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

in which T is the periodic time, and m the mean daily motion of a point on the radius vector at the unit's distance from the sun; whence we have

$$m\,dt=d\alpha-2e\cos V dV+\tfrac{3}{4}e^2\cos 2V d2V-\&c.;$$

and by integration,

$$mt+C=\alpha-2e\sin V+\tfrac{3}{4}e^2\sin 2V-\&c.$$

Making $V=0$, and estimating α from the line through the vernal equinox, we have

$$mt_p+C=\alpha_p;$$

in which α_p is the longitude of the perihelion, and t_p the time from perihelion passage. Whence, by subtraction,

$$m\,(t - t_p) = \alpha - \alpha_p - 2\,e \sin V + \tfrac{3}{4}\,e^2 \sin 2\,V - \&c. \quad . \quad (e)$$

but

$$\alpha - \alpha_p = V;$$

whence

$$m\,(t - t_p) = V - 2\,e \sin V + \tfrac{3}{4}\,e^2 \sin 2\,V - \&c. \quad . \quad . \quad (g)$$

in which $m\,(t - t_p)$ is the mean anomaly, being the mean angular distance from perihelion.

Adding α_p to each member of Eq. (e), making

$$m\,(t - t_p) + \alpha_p = \alpha_m,$$

and writing $\alpha - \alpha_p$ for V, we find

$$\alpha_m = \alpha - 2\,e \sin\,(\alpha - \alpha_p) + \tfrac{3}{4}\,e^2 \sin 2\,(\alpha - \alpha_p) - \&c. \quad . \quad . \quad (h)$$

in which α_m is the mean, and α the true longitude.

Denote by L the mean longitude at any given epoch, say the beginning of the year, and by t the interval of time since the epoch; then will

$$\alpha_m = L + m\,t,$$

and

$$L + m\,t = \alpha - 2\,e \sin\,(\alpha - \alpha_p) + \tfrac{3}{4}\,e^2 \,.\, \sin 2\,(\alpha - \alpha_p) - \&c. \, . \, . \quad (i)$$

Again, assuming

$$\cos V = \frac{\cos u - e}{1 - e \cos u} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (j)$$

and substituting in Eq. (f), and replacing the first factor by its value in Eq. (d), we have

$$m t + C = \int d u\,(1 - e \cos u) = u - e \sin u;$$

and making $t = t_p$, in which case the body is in perihelion, where $V = 0$ and therefore $u = 0$, we have

$$m\,t_p + C = 0;$$

and by subtraction, making $t - t_p = t'$,

$$m t' = u - e \sin u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (k)$$

From Eq. (j) we have

$$\tan \frac{V}{2} = \sqrt{\frac{1 + e}{1 - e}} \,.\, \tan \frac{u}{2};$$

from which we have

$$V = u + e \sin u + \frac{e^2}{4} \cdot \sin 2u \quad . \quad . \quad . \quad . \quad . \quad (l)$$

and from Eq. (k),

$$u = mt' + 2e \sin mt' + \tfrac{1}{2} e^2 \sin 2mt' + \&c. \quad . \quad . \quad . \quad (m)$$

which substituted in Eq. (l) gives

$$V = mt' + 2e \sin mt' + \tfrac{5}{4} e^2 \cdot \sin 2mt' \quad . \quad . \quad . \quad . \quad (n)$$

whence

$$V - mt' = 2e \sin mt' + \tfrac{5}{4} e^2 \cdot \sin 2mt' \quad . \quad . \quad . \quad . \quad (o)$$

The first member, which is the difference between the true and mean anomalies, is called the *equation of the centre.* It is expressed in terms of the eccentricity and mean anomaly. The auxiliary angle u is called the *eccentric anomaly.*

APPENDIX VI.

PLANETS' ELEMENTS.

Differentiating the equation

$$e \cos v = \frac{L}{r} - 1,$$

we have, after dividing by dt,

$$e \sin v \cdot \frac{dv}{dt} = \frac{L}{r^2} \cdot \frac{dr}{dt};$$

but, *Analytical Mechanics*, § 192,

$$\frac{dv}{dt} = \frac{2c}{r^2};$$

which substituted above gives, after making

$$\frac{dr}{dt} = V_r,$$

$$e \sin v = \frac{L}{2c} \cdot V_r;$$

which is Eq. (100) of the text.

APPENDIX VII.

PLANETS' ELEMENTS.

From Eq. (277), *Analytical Mechanics*, we have

$$a\,(1-e^2)=\frac{4\,c^2}{k};$$

whence, making $\mu=k$,

$$2\,c=\sqrt{\mu}\,.\,\sqrt{a\,(1-e^2)};$$

and this in the equation

$$\frac{d\,v}{d\,t}=\frac{2\,c}{r^2},$$

Appendix VI., gives

$$\frac{d\,v}{d\,t}=\frac{\sqrt{\mu}\,.\,\sqrt{a\,(1-e^2)}}{r^2};$$

and substituting the value of r^2 from the equation

$$r=\frac{a\,(1-e^2)}{1+e\cos v},$$

we find

$$d\,t=\frac{a^{\frac{3}{2}}\,.\,(1-e^2)^{\frac{3}{2}}}{\sqrt{\mu}}\,.\,\frac{d\,v}{(1+e\cos v)^2}.$$

To integrate this, assume

$$\cos v=\frac{\cos u-e}{1-e\cos u};$$

from which find the value of $d\,v$; eliminate $d\,v$ and $\cos v$ above, and we have,

$$d\,t=\frac{a^{\frac{3}{2}}}{\sqrt{\mu}}\,.\,(1-e\cos u)\,d\,u;$$

and by integration

$$t+C=\frac{a^{\frac{3}{2}}}{\sqrt{\mu}}\,(u-e\sin u).$$

But, *Analytical Mechanics*, § 201,

$$\frac{a^{\frac{3}{2}}}{\sqrt{\mu}}=\frac{T}{2\pi},$$

in which T is the periodic time.

Whence, making $t = 0$ when $u = 0$, we have $C = 0$, and

$$\frac{2\pi}{T} \cdot t = u - e \sin u;$$

and denoting the mean motion by n, we have

$$n = \frac{2\pi}{T};$$

and finally

$$nt = u - e \sin u;$$

which is Eq. (106) of the text.

The quantity t is the time from perihelion, for by making $u = 0$, we have

$$t = 0; \quad \cos v = 1, \text{ or } v = 0°.$$

APPENDIX VIII.

PLANETS' ELEMENTS.

Differentiate the equation

$$r^2 = x^2 + y^2 + z^2,$$

and divide by $2rdt$, we have

$$\frac{dr}{dt} = \frac{x}{r} \cdot \frac{dx}{dt} + \frac{y}{r} \cdot \frac{dy}{dt} + \frac{z}{r} \cdot \frac{dz}{dt}$$

and making

$$\frac{dr}{dt} = V_r; \quad \frac{dx}{dt} = V_x; \quad \frac{dy}{dt} = V_y; \quad \frac{dz}{dt} = V_z;$$

we have

$$V_r = \frac{x}{r} \cdot V_x + \frac{y}{r} \cdot V_y + \frac{z}{r} \cdot V_z,$$

which is Eq. (112) of the text.

APPENDIX IX.

PLANETS ELEMENTS.

Make

α_1, α_2, α_3, the observed right ascensions;
β_1, β_2, β_3, the observed north polar distances;
t_1, t_2, t_3, the mean times of observations reduced to any first meridian, say that of Greenwich;

and suppose the observed quantities corrected to the mean equinox and mean position of the equator at the beginning of the year.

In the interval of time required for light to travel from a roaming body to the earth, the body describes some definite portion of its path, and at any given instant we see the place it left and not that which it actually occupies. We look, as it were, at luminous places on the orbit, but always behind the body's true place. The position which a body occupied at the instant the light started, and in which it is seen at a given time, is called its *virtual* place at that time; and that which it actually occupies is called its *true* place.

Conceive three sets of parallel rectangular co-ordinate axes, one set through the place of observation, another through the centre of the earth, and the third through the centre of the sun. Take the planes xy parallel to the plane of the equinoctial, the axes of x parallel to the line of the equinoxes and positive towards the first point of *Aries*.

Denote by $\rho_{\prime}$ the distance of the body's virtual place from the earth at the time t_1, and by ν the time required for light to travel over the mean radius of the earth's orbit, which we have taken as unity; then will $\nu\rho_{\prime}$ be the time required for light to travel over the distance $\rho_{\prime}$.

Denote by $\bar{x}$, $\bar{y}$, $\bar{z}$ the co-ordinates of the virtual, and x, y, z the co-ordinates of the true place of the body at the time t_1, referred to the centre of the earth; then, regarding the motion of the body as uniform during the time $\nu\rho_{\prime}$, will

$$\left.\begin{aligned} \bar{x} &= x - \nu\rho_{\prime} \cdot \frac{dx}{dt} \\ \bar{y} &= y - \nu\rho_{\prime} \cdot \frac{dy}{dt} \\ \bar{z} &= z - \nu\rho_{\prime} \cdot \frac{dz}{dt} \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

Denote the co-ordinates of the sun, cleared of aberration at the time t_1, and referred to the same origin, by $X_{,}$, $Y_{,}$, $Z_{,}$; and the heliocentric co ordinates of the true place of the body at the same time by $x_{,}$, $y_{,}$, $z_{,}$; then will

$$x = X_{,} + x_{,},$$
$$y = Y_{,} + y_{,},$$
$$z = Z_{,} + z_{,};$$

which in Eqs. (1) give

$$\left.\begin{aligned} \bar{x} &= X_{,} + x_{,} - \nu \rho_{,} \cdot \frac{d\,(X_{,} + x_{,})}{dt} \\ y &= Y_{,} + y_{,} - \nu \rho_{,} \cdot \frac{d\,(Y_{,} + y_{,})}{dt} \\ \bar{z} &= Z_{,} + z_{,} - \nu \rho_{,} \cdot \frac{d\,(Z_{,} + z_{,})}{dt} \end{aligned}\right\} \quad \ldots\ldots \quad (2)$$

in which

$$\rho_{,} = (z_{,} + Z_{,}) \sec \beta_1 \quad \ldots\ldots\ldots \quad (3)$$

or, which may be preferable, if the body be near the equator,

$$\rho_{,} = (x_{,} + X_{,}) \sec \alpha_1 \operatorname{cosec} \beta_1 \ldots\ldots\ldots \quad (4)$$

Denoting the co-ordinates of the virtual place of the body at the time t_1, referred to the place of observation, by x', y', z'; and the co-ordinates at the same time of the place of observation, referred to the centre of the earth, by $f_{,}$, $g_{,}$, and h; then will

$$\bar{x} = x' + f_{,},$$
$$\bar{y} = y' + g_{,},$$
$$\bar{z} = z' + h;$$

which substituted in Eqs. (2) give

$$\left.\begin{aligned} x' &= x_{,} + X_{,} - \rho_{,}\nu \left(\frac{dx_{,}}{dt} + \frac{dX_{,}}{dt}\right) - f_{,} \\ y' &= y_{,} + Y_{,} - \rho_{,}\nu \left(\frac{dy_{,}}{dt} + \frac{dY_{,}}{dt}\right) - g_{,} \\ z' &= z_{,} + Z_{,} - \rho_{,}\nu \left(\frac{dz_{,}}{dt} + \frac{dZ_{,}}{dt}\right) - h \end{aligned}\right\} \quad \ldots\ldots \quad (5)$$

But

$$\left.\begin{aligned} y' - x' \tan \alpha_1 &= 0 \\ z' - x' \tan \theta_1 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (6)$$

in which $\cot \theta_1 = \cos \alpha_1 \cdot \tan \beta_1$ . . . . (7)

Also, if l denote the geocentric colatitude of the place of observation, ρ the corresponding radius of the earth, and $T_{,}$ the sidereal time of observation reduced to degrees, then will

$$\left.\begin{aligned} f_{,} &= \rho \cdot \sin l \cdot \cos T_{,} \\ g_{,} &= \rho \cdot \sin l \cdot \sin T_{,} \\ h &= \rho \cdot \cos l \,. \end{aligned}\right\} \quad \ldots \ldots \ldots (8)$$

and

$$\rho = \frac{\textit{sun's horizontal parallax at the place of observation}}{\textit{number of seconds in an arc equal in length to radius}} \quad \ldots (9)$$

Multiplying the first of Eqs. (5) by $\tan \alpha_1$ and subtracting the product from the second, then by $\tan \theta_1$ and subtracting the product from the third, and reducing by the relations of Eqs. (6), we have

$$y_1 - x_1 \tan \alpha_1 = (X_1 - f_1) \tan \alpha_1 - Y_1 + g_1 + \nu\rho_1 \left[\frac{dy_1}{dt} + \frac{dY_1}{dt} - \tan \alpha_1 \left(\frac{dx_1}{dt} + \frac{dX_1}{dt}\right)\right]$$

$$z_1 - x_1 \tan \theta_1 = (X_1 - f_1) \tan \theta_1 - Z_1 + h + \nu\rho_1 \left[\frac{dz_1}{dt} + \frac{dZ_1}{dt} - \tan \theta_1 \left(\frac{dx_1}{dt} + \frac{dX_1}{dt}\right)\right]$$

in like manner

$$y_2 - x_2 \tan \alpha_2 = (X_2 - f_2) \tan \alpha_2 - Y_2 + g_2 + \nu\rho_2 \left[\frac{dy_2}{dt} + \frac{dY_2}{dt} - \tan \alpha_2 \left(\frac{dx_2}{dt} + \frac{dX_2}{dt}\right)\right]$$

$$z_2 - x_2 \tan \theta_2 = (X_2 - f_2) \tan \theta_2 - Z_2 + h + \nu\rho_2 \left[\frac{dz_2}{dt} + \frac{dZ_2}{dt} - \tan \theta_2 \left(\frac{dx_2}{dt} + \frac{dX_2}{dt}\right)\right] \quad (10)$$

and

$$y_3 - x_3 \tan \alpha_3 = (X_3 - f_3) \tan \alpha_3 - Y_3 + g_3 + \nu\rho_3 \left[\frac{dy_3}{dt} + \frac{dY_3}{dt} - \tan \alpha_3 \left(\frac{dx_3}{dt} + \frac{dX_3}{dt}\right)\right]$$

$$z_3 - x_3 \tan \theta_3 = (X_3 - f_3) \tan \theta_3 - Z_3 + h + \nu\rho_3 \left[\frac{dz_3}{dt} + \frac{dZ_3}{dt} - \tan \theta_3 \left(\frac{dx_3}{dt} + \frac{dX_3}{dt}\right)\right]$$

in which, as in equations (3) and (4),

$$\left.\begin{aligned} \rho_2 &= (z_2 + Z_2) \sec \beta_2 \\ \rho_3 &= (z_3 + Z_3) \sec \beta_3 \end{aligned}\right\} \quad \ldots \ldots (11)$$

or, if the body be near the equator,

$$\left.\begin{aligned} \rho_2 &= (x_2 + X_2) \sec \alpha_2 \cdot \operatorname{cosec} \beta_2 \\ \rho_3 &= (x_3 + X_3) \sec \alpha_3 \cdot \operatorname{cosec} \beta_3 \end{aligned}\right\} \quad \ldots \ldots (12)$$

Now make

$$t_1 = t_2 - \tau, \text{ and } t_3 = t_2 + \tau' ;$$

then, because x_1 and x_3 are functions of t, which become x_2 when τ and τ' become zero, we have by Taylor's formula,

$$\left.\begin{array}{l} x_1 = x_2 - \dfrac{d\,x_2}{d\,t} \cdot \tau + \dfrac{d^2\,x_2}{d\,t^2} \cdot \dfrac{\tau^2}{2} - \dfrac{d^3 x_2}{d\,t^3} \cdot \dfrac{\tau^3}{6} + \dfrac{d^4 x_2}{d\,t^4} \cdot \dfrac{\tau^4}{24} - \text{\&c.} \\ \text{and the same for } y_1 \text{ and } z_1; \\ x_3 = x_2 + \dfrac{d\,x_2}{d\,t} \cdot \tau' + \dfrac{d^2 x_2}{d\,t^2} \cdot \dfrac{\tau'^2}{2} + \dfrac{d^3 x_2}{d\,t^3} \cdot \dfrac{\tau'^3}{6} + \dfrac{d^4 x_2}{d\,t^4} \cdot \dfrac{\tau'^4}{24} + \text{\&c.} \\ \text{and the same for } y_3 \text{ and } z_3. \end{array}\right\} \quad (13)$$

The intervals τ and τ' must be such as to make these expressions converge rapidly, and it will rarely if ever be necessary to retain the terms of the series involving powers of τ and τ' higher than the fourth.

Denote by μ the acceleration due to the sun's attractive force at the mean distance of the earth, and by r_2 the distance of the body from the sun at the time t_2, then will $\dfrac{\mu}{r_2^2}$ be the acceleration due to the sun's attraction on the body, and we shall have

$$\left.\begin{array}{l} \dfrac{d^2 x_2}{d\,t^2} = - \dfrac{\mu}{r_2^2} \cdot \dfrac{x_2}{r_2} = - \dfrac{\mu\,x_2}{r_2^3} \\ \dfrac{d^2 y_2}{d\,t^2} = - \dfrac{\mu}{r_2^2} \cdot \dfrac{y_2}{r_2} = - \dfrac{\mu\,y_2}{r_2^3} \\ \dfrac{d^2 z_2}{d\,t^2} = - \dfrac{\mu}{r_2^2} \cdot \dfrac{z_2}{r_2} = - \dfrac{\mu\,z_2}{r_2^3} \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; (14)$$

Differentiating and dividing by $d\,t$, twice, we have

$$\frac{d^3 x_2}{d\,t^3} = - \frac{\mu}{r_2^3} \cdot \frac{d\,x_2}{d\,t} + \frac{3\,\mu}{r_2^4} \cdot \frac{d\,r_2}{d\,t} \cdot x_2,$$

$$\frac{d^4 x_2}{d\,t^4} = \frac{\mu^2}{r_2^6} \cdot x_2 + \frac{6\,\mu}{r_2^4} \cdot \frac{d\,r_2}{d\,t} \cdot \frac{d\,x_2}{d\,t} + \frac{3\,\mu}{r_2^4} \cdot \frac{d^2 r_2}{d\,t^2} \cdot x_2 - \frac{12\,\mu}{r_2^5}\;\frac{d\,r_2^2}{d\,t^2} \cdot x_2\,;$$

and the same for

$$\frac{d^3 y_2}{d\,t^3},\quad \frac{d^4 y_2}{d\,t^4},\quad \frac{d^3 z_2}{d\,t^3} \text{ and } \frac{d^4 z_2}{d\,t^4},$$

by simply writing y and z successively for x.

These values being substituted in Eqs. (13), and the resulting values of

x_1, y_1, z_1, x_3, y_3, and z_3;

and also those of

$$\frac{dx_1}{dt}, \quad \frac{dy_1}{dt}, \quad \frac{dz_1}{dt}, \quad \frac{dx_3}{dt}, \quad \frac{dy_3}{dt}, \text{ and } \frac{dz_3}{dt},$$

obtained therefrom in Eqs. (10)—observing to limit the series to the second order of differentials in the aberration terms, or those of which ν is a factor, and to the fourth order in the others—we have, after making

$$\left.\begin{aligned}
A_1 &= (X_1 - f_1)\tan\alpha_1 - Y_1 + g_1 \\
B_1 &= (X_1 - f_1)\tan\theta_1 - Z_1 + h \\
A_2 &= (X_2 - f_2)\tan\alpha_2 - Y_2 + g_2 \\
B_2 &= (X_2 - f_2)\tan\theta_2 - Z_2 + h \\
A_3 &= (X_3 - f_3)\tan\alpha_3 - Y_3 + g_3 \\
B_3 &= (X_3 - f_3)\tan\theta_3 - Z_3 + h
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\left.\begin{aligned}
k_1 &= \frac{dY_1}{dt} - \tan\alpha_1 \frac{dX_1}{dt} \\
l_1 &= \frac{dZ_1}{dt} - \tan\theta_1 \frac{dX_1}{dt} \\
k_2 &= \frac{dY_2}{dt} - \tan\alpha_2 \frac{dX_2}{dt} \\
l_2 &= \frac{dZ_2}{dt} - \tan\theta_2 \frac{dX_2}{dt} \\
k_3 &= \frac{dY_3}{dt} - \tan\alpha_3 \frac{dX_3}{dt} \\
l_3 &= \frac{dZ_3}{dt} - \tan\theta_3 \frac{dX_3}{dt}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\left.\begin{aligned}
U &= \frac{\mu\tau^2}{2r_2^3} + \frac{\mu\tau^3}{2r_2^4}\cdot\frac{dr_2}{dt} + \frac{\mu\tau^4}{2r_2^5}\cdot\frac{dr_2^2}{dt^2} - \frac{\mu\tau^4}{8r_2^4}\cdot\frac{d^2r_2}{dt^2} - \frac{\mu^2\tau^4}{24r_2^6} \\
W &= -\frac{\mu\tau^3}{6r_2^3} - \frac{\mu\tau^4}{4r_2^4}\cdot\frac{dr_2}{dt} \\
U' &= \frac{\mu\tau'^2}{2r_2^3} - \frac{\mu\tau'^3}{2r_2^4}\cdot\frac{dr_2}{dt} + \frac{\mu\tau'^4}{2r_2^5}\cdot\frac{dr_2^2}{dt^2} - \frac{\mu\tau'^4}{8r_2^4}\cdot\frac{d^2r_2}{dt^2} - \frac{\mu^2\tau'^4}{24r_2^6} \\
W' &= \frac{\mu\tau'^3}{6r_2^3} - \frac{\mu\tau'^4}{4r_2^4}\cdot\frac{dr_2}{dt}
\end{aligned}\right\} \quad (17)$$

$$\left.\begin{aligned}
a_1 &= \left(U+\frac{\mu\nu\rho_1\tau}{r_2^3}\right)(y_2-x_2\tan\alpha_1)+(W+\nu\rho_1)\left(\frac{dy_2}{dt}-\tan\alpha_1\frac{dx_2}{dt}\right)+\nu\rho_1 k_1\\
b_1 &= \left(U+\frac{\mu\nu\rho_1\tau}{r_2^3}\right)(z_2-x_2\tan\theta_1)+(W+\nu\rho_1)\left(\frac{dz_2}{dt}-\tan\theta_1\frac{dx_2}{dt}\right)+\nu\rho_1 l_1\\
a_2 &= \nu\rho_2\left(k_2+\frac{dy_2}{dt}-\tan\alpha_2\frac{dx_2}{dt}\right)\\
b_2 &= \nu\rho_2\left(l_2+\frac{dz_2}{dt}-\tan\theta_2\frac{dx_2}{dt}\right)\\
a_3 &= \left(U'-\frac{\mu\nu\rho_3\tau'}{r_2^3}\right)(y_2-x_2\tan\alpha_3)+(W'+\nu\rho_3)\left(\frac{dy_2}{dt}-\tan\alpha_3\frac{dx_2}{dt}\right)+\nu\rho_3 k_3\\
b_3 &= \left(U'-\frac{\mu\nu\rho_3\tau'}{r_2^3}\right)(z_2-x_2\tan\theta_3)+(W'+\nu\rho_3)\left(\frac{dz_2}{dt}-\tan\theta_3\frac{dx_2}{dt}\right)+\nu\rho_3 l_3
\end{aligned}\right\}(18)$$

and in which, by substituting the values of z_1 and z_3, x_1 and x_3 in equations (3), (4), (11), and (12),

$$\left.\begin{aligned}
\rho_1 &= \left[z_2\left(1-\frac{\mu\tau^2}{2r_2^3}\right)-\frac{dz_2}{dt}.\tau+Z_1\right]\sec\beta_1\\
\rho_2 &= (z_2+Z_2)\sec\beta_2\\
\rho_3 &= \left[z_2\left(1-\frac{\mu\tau'^2}{2r_2^3}\right)+\frac{dz_2}{dt}.\tau'+Z_3\right]\sec\beta_3
\end{aligned}\right\}\quad . \; . \; . \; . \; (19)$$

or

$$\left.\begin{aligned}
\rho_1 &= \left[x_2\left(1-\frac{\mu\tau^2}{2r_2^3}\right)-\frac{dx_2}{dt}.\tau+X_1\right]\sec\alpha_1 . \operatorname{cosec}\beta_1\\
\rho_2 &= (x_2+X_2)\sec\alpha_2 . \operatorname{cosec}\beta_2\\
\rho_3 &= \left[x_2\left(1-\frac{\mu\tau'^2}{2r_2^3}\right)+\frac{dx_2}{dt}.\tau'+X_3\right]\sec\alpha_3 . \operatorname{cosec}\beta_3
\end{aligned}\right\}\quad . \; (20)$$

we obtain

$$\left.\begin{aligned}
y_2-x_2\tan\alpha_1-\frac{dy_2}{dt}\tau+\tan\alpha_1\frac{dx_2}{dt}\tau &= A_1+a_1\\
z_2-x_2\tan\theta_1-\frac{dz_2}{dt}\tau+\tan\theta_1\frac{dx_2}{dt}\tau &= B_1+b_1\\
y_2-x_2\tan\alpha_2 &= A_2+a_2\\
z_2-x_2\tan\theta_2 &= B_2+b_2\\
y_2-x_2\tan\alpha_3+\frac{dy_2}{dt}\tau'-\tan\alpha_3\frac{dx_2}{dt}\tau' &= A_3+a_3\\
z_2-x_2\tan\theta_3+\frac{dz_2}{dt}\tau'-\tan\theta_3\frac{dx_2}{dt}\tau' &= B_3+b_3
\end{aligned}\right\}\quad . \; . \; . \; (21)$$

Regarding x_2, y_2, z_2, $\frac{d x_2}{d t}$, $\frac{d y_2}{d t}$, and $\frac{d z_2}{d t}$ as unknown quantities, we obtain by elimination,

$$x_2 = (P + p) - (Q + q) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

in which, by making

$$\left.\begin{aligned} R &= \left(1 + \frac{\tau'}{\tau}\right) \cdot \frac{\sin(\alpha_2 - \alpha_1)\cos\alpha_3}{\sin(\alpha_1 - \alpha_3)\cos\alpha_2} \\ S &= \left(1 + \frac{\tau'}{\tau}\right) \cdot \frac{\sin(\theta_3 - \theta_1)\cos\theta_3}{\sin(\theta_1 - \theta_3)\cos\theta_2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (23)$$

we have

$$\left.\begin{aligned} P &= \frac{\cos\alpha_1\cos\alpha_3}{(R - S)\sin(\alpha_1 - \alpha_3)} \cdot \left[A_1\frac{\tau'}{\tau} + A_3 - A_2\left(1 + \frac{\tau'}{\tau}\right)\right] \\ Q &= \frac{\cos\theta_1\cos\theta_3}{(R - S)\sin(\theta_1 - \theta_3)} \cdot \left[B_1\frac{\tau'}{\tau} + B_3 - B_2\left(1 + \frac{\tau'}{\tau}\right)\right] \\ p &= \frac{\cos\alpha_1\cos\alpha_3}{(R - S)\sin(\alpha_1 - \alpha_3)} \cdot \left[a_1\frac{\tau'}{\tau} + a_3 - a_2\left(1 + \frac{\tau'}{\tau}\right)\right] \\ q &= \frac{\cos\theta_1\cos\theta_3}{(R - S)\sin(\theta_1 - \theta_3)} \cdot \left[b_1\frac{\tau'}{\tau} + b_3 - b_2\left(1 + \frac{\tau'}{\tau}\right)\right] \end{aligned}\right\} (24)$$

or by making

$$D = \frac{\cos\alpha_1\cos\alpha_3}{(R - S)\sin(\alpha_1 - \alpha_3)}; \qquad E = D\frac{\tau'}{\tau};$$

$$F = \frac{\cos\theta_1\cos\theta_3}{(R - S)\sin(\theta_1 - \theta_3)}; \qquad G = F\frac{\tau'}{\tau};$$

we have

$$\left.\begin{aligned} P &= D(A_3 - A_2) + E(A_1 - A_2) \\ Q &= F(B_3 - B_2) + G(B_1 - B_2) \\ p &= D(a_3 - a_2) + E(a_1 - a_2) \\ q &= F(b_3 - b_3) + G(b_1 - b_2) \end{aligned}\right\} \quad . \quad . \quad . \quad (25)$$

and making

$$C = P - Q,$$

we have

$$\left.\begin{aligned}
x_2 &= C + D(a_3 - a_2) + E(a_1 - a_2) + F(b_2 - b_3) + G(b_2 - b_1) \\
y_2 &= x_2 \tan \alpha_2 + A_2 + a_2 \\
z_2 &= x_2 \tan \theta_2 + B_2 + b_2 \\
\frac{dx_2}{dt} &= \frac{1}{\tau'}(R - S)(P + p) - \frac{x_2}{\tau'}(1 + R) \\
\frac{dy_2}{dt} &= \frac{dx_2}{dt} \tan \alpha_1 - \frac{1}{\tau}(x_2 \tan \alpha_1 + A_1 + a_1 - y_2) \\
\frac{dz_2}{dt} &= \frac{dx_2}{dt} \tan \theta_1 - \frac{1}{\tau}(x_2 \tan \theta_1 + B_1 + b_1 - z_2) \\
&\text{or instead of the last two,} \\
\frac{dy_2}{dt} &= \frac{dx_2}{dt} \tan \alpha_3 + \frac{1}{\tau'}(x_2 \tan \alpha_3 + A_3 + a_3 - y_2) \\
\frac{dz_2}{dt} &= \frac{dx_2}{dt} \tan \theta_3 + \frac{1}{\tau'}(x_2 \tan \theta_3 + B_3 + b_3 - z_2)
\end{aligned}\right\} \quad (26)$$

Now although the Eqs. (26) express the values of the co-ordinates and components of the velocity of the body at the time of the second observation, they involve the geocentric distances ρ_1, ρ_2, ρ_3, and the radius vector r_2, which are unknown, and the solution of the problem can only be accomplished by successive approximations.

First Approximation.

Let us first neglect the terms involving aberration, and those containing as factors powers of τ and τ' higher than the second. This will give, Eqs. (17),

$$U = \frac{\mu \tau^2}{2 r_2^3}; \qquad W = 0; \qquad U' = \frac{\mu \tau'^2}{2 r_2^3}; \qquad W' = 0;$$

and Eqs. (18),

$$\left.\begin{aligned}
a_1 &= \frac{\mu \tau^2}{2 r_2^3}(y_2 - x_2 \tan \alpha_1) \\
b_1 &= \frac{\mu \tau^2}{2 r_2^3}(z_2 - x_2 \tan \theta_1) \\
a_2 &= 0 \qquad b_2 = 0 \\
a_3 &= \frac{\mu \tau'^2}{2 r_2^3}(y_2 - x_2 \tan \alpha_3) \\
b_3 &= \frac{\mu \tau'^2}{2 r_2^3}(z_2 - x_2 \tan \theta_3)
\end{aligned}\right\} \quad \ldots\ldots (27)$$

and as all terms involving powers of τ and τ' higher than the second are to be neglected we obtain from Eqs. (21), for the values of the severa. factors above,

$$\left.\begin{aligned} y_2 - x_2 \tan \alpha_1 &= A_1 \\ z_2 - x_2 \tan \theta_1 &= B_1 \\ y_2 - x_2 \tan \alpha_2 &= A_2 \\ z_2 - x_2 \tan \theta_2 &= B_2 \\ y_2 - x_2 \tan \alpha_3 &= A_3 \\ z_2 - x_2 \tan \theta_3 &= B_3 \end{aligned}\right\} \quad \ldots \ldots \quad (28)$$

which substituted in Eqs. (27) give

$$\left.\begin{aligned} a_1 &= \frac{A_1 \mu \tau^2}{2 r_2^3}; & b_1 &= \frac{B_1 \mu \tau^2}{2 r_2^3}; \\ a_2 &= 0; & b_2 &= 0; \\ a_3 &= \frac{A_3 \mu \tau'^2}{2 r_2^3}; & b_3 &= \frac{B_3 \mu \tau'^2}{2 r_2^3}; \end{aligned}\right\} \quad \ldots \quad (29)$$

and these in the first of Eqs. (26) give

$$x_2 = C + \frac{\mu \tau^2}{2} \left(D A_3 \frac{\tau'^2}{\tau^2} + E A_1 - F B_3 \frac{\tau'^2}{\tau^2} - G B_1 \right) . \frac{1}{r_2^3};$$

and making

$$N = \frac{\mu \tau^2}{2} \left(D A_3 \frac{\tau'^2}{\tau^2} + E A_1 - F B_3 \frac{\tau'^2}{q^2} - G B_1 \right) \quad . \; . \; (30)$$

we have

$$x_2 = C + \frac{N}{r_2^3} \quad \ldots \ldots \ldots \quad (31)$$

and this value of x_2, and the foregoing values of a_2 and b_2 in the second and third of Eqs. (26), give

$$y_2 = C \tan \alpha_2 + A_2 + \frac{N \tan \alpha_2}{r_2^3};$$

$$z_2 = C \tan \theta_2 + B_2 + \frac{N \tan \theta_2}{r_2^3};$$

or making

$$\left.\begin{aligned} C' &= C \tan \alpha_2 + A_2; & N' &= N \tan \alpha_2 \\ C'' &= C \tan \theta_2 + B_2; & N'' &= N \tan \theta_2 \end{aligned}\right\} \quad \ldots \quad (32)$$

we have
$$y_2 = C' + \frac{N'}{r_2^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$z_2 = C'' + \frac{N''}{r_2^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

and hence

$$r_2^2 = \left(C + \frac{N}{r_2^3}\right)^2 + \left(C' + \frac{N'}{r_2^3}\right)^2 + \left(C'' + \frac{N''}{r_2^3}\right)^2 . \quad . \quad . \quad (35)$$

This equation must be solved tentatively. If the body be considerably more distant than the earth from the sun, C, C', C'' are respectively approximations to the values of x_2, y_2, z_2; and if considerably less distant, then $\frac{N}{r_2^3}$, $\frac{N'}{r_2^3}$, and $\frac{N''}{r_2^3}$ are approximations to the same quantities. It is evident that a value for r_2 must be selected which will make the greatest values of $C + \frac{N}{r_2^3}$, $C' + \frac{N'}{r_2^3}$, and $C'' + \frac{N''}{r_2^3}$, without regard to signs less than r_2 and greater than $\frac{r_2}{\sqrt{3}}$. And if $C + \frac{N}{r_2^3}$, for instance, be the greatest, the value of r_2 which satisfies the equation

$$C + \frac{N}{r_2^3} = \frac{r_2}{2} \cdot \left(1 + \frac{1}{\sqrt{3}}\right),$$

will differ from the true value by a quantity less than $\frac{r_2}{2}\left(1 - \frac{1}{\sqrt{3}}\right)$, that is, less than $0.21\, r_2$.

But the solution of Eq. (35) may be greatly facilitated by means of an elegant geometrical construction, due to J. J. Waterston, Esq. Thus:

Squaring the terms as indicated in the second member, recalling the relation in Eq. (7), which will give

$$1 + \tan^2 \alpha_2 + \tan^2 \theta_2 = \tan^2 \theta_2 \sec^2 \beta_2;$$

substituting the values of C', C'', N', N'' in Eqs. (32), and making

$$\begin{aligned} r &= N \,.\, \tan \theta_2 \sec \beta_2; \\ K &= A_2 \tan \alpha_2 \,.\, \cot \theta_2 \cos \beta_2 + B_2 \cos \beta_2; \\ \beta &= K + C \tan \theta_2 \sec \beta_2; \\ \alpha^2 &= A_2^2 + B_2^2 - K^2; \end{aligned}$$

equation (35) may be written

$$r_2^2 = \alpha^2 + \left(\beta + \frac{r}{r_2^2}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (36)$$

or
$$\frac{r_2^2}{a^2} - 1 = \left(\frac{\beta}{\alpha} + \frac{r}{\alpha^4} \cdot \frac{\alpha^3}{r_2^3}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Make
$$\frac{r_2}{\alpha} = \operatorname{cosec} \theta,$$

then will
$$\frac{r_2^2}{\alpha^2} - 1 = \cot^2 \theta, \quad \text{and} \quad \frac{\alpha^3}{r_2^3} = \sin^3 \theta;$$

and Eq. (37),
$$\cot \theta = \frac{\beta}{\alpha} + \frac{r}{\alpha^4} \cdot \sin^3 \theta;$$

and making $\cot \theta = y$, and $\sin^3 \theta = x$,
$$y = \frac{\beta}{\alpha} + \frac{r}{\alpha^4} x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Also
$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta = \frac{1}{\sin^2 \theta},$$

or
$$1 + y^2 = \frac{1}{x^{\frac{2}{3}}};$$

whence
$$x = \frac{1}{(1 + y^2)^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

If the curve of which this is the equation be described graphically, and the straight line, of which (38) is the equation, be drawn, the abscissa of their point of intersection will give the value of $\sin^3 \theta$, or $\frac{\alpha^3}{r_2^3}$; and r_2 becomes known. Its value may be verified by substitution in Eq. (35).

This value of r_2 in Eq. (31) gives x_2, and this in the second and third of Eqs. (26) gives y_2 and z_2; also, r_2 in Eqs. (29) gives a_1, b_1, a_3, b_3, and these in the third of (25) will give p, which with a_1 and b_1 in fourth, fifth, and sixth, or fourth, seventh, and eighth of (26), give $\frac{dx_2}{dt}$, $\frac{dy_2}{dt}$, and $\frac{dz_2}{dt}$, and the values of ρ_1, ρ_2, ρ_3 in Eqs. (19) or (20).

Second Approximation.

By differentiating the equation
$$r_2^2 = x_2^2 + y_2^2 + z_2^2,$$

and dividing by $r_2\, dt$, we have
$$\frac{dr_2}{dt} = \frac{x_2}{r_2} \cdot \frac{dx_2}{dt} + \frac{y_2}{r_2} \cdot \frac{dy_2}{dt} + \frac{z_2}{r_2} \cdot \frac{dz_2}{dt} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

The first term of this equation becoming thus known, the values of U, W, U', and W', Eqs. (17), may be computed to include the third powers of τ and τ', then the corresponding values of a_1, b_1, a_2, b_2, a_3, b_3, Eqs. (18). Then, denoting by Δa_1, Δb_1, Δa_2, Δb_2, Δa_3, Δb_3 the difference between the first and second values of the quantities written after the symbol Δ, and observing a like notation for the other quantities, we have for computing the first corrections to x_2, y_2, z_2, $\frac{dx_2}{dt}$, $\frac{dy_2}{dt}$, and $\frac{dz_2}{dt}$ from the third and fourth of Eqs. (25),

$$\left.\begin{aligned}\Delta p &= D\,(\Delta a_3 - \Delta a_2) + E\,(\Delta a_1 - \Delta a_2)\\ \Delta q &= F\,(\Delta b_3 - \Delta b_2) + G\,(\Delta b_1 - \Delta b_2)\end{aligned}\right\} \quad . \quad . \quad . \quad (41)$$

and then from Eqs. (26),

$$\left.\begin{aligned}\Delta x_2 &= \Delta p - \Delta q\\ \Delta y_2 &= \Delta x_2 \tan \alpha_2 + \Delta a_2\\ \Delta z_2 &= \Delta x_2 \tan \theta_2 + \Delta b_2\\ \Delta \frac{dx_2}{dt} &= \frac{\Delta p}{\tau'}(R - S) - \frac{\Delta x_2}{\tau'}(1 + R)\\ \Delta \frac{dy_2}{dt} &= \Delta \frac{dx_2}{dt} \tan \alpha_1 - \frac{1}{\tau}(\Delta x_2 \tan \alpha_1 + \Delta a_1 - \Delta y_2)\\ \Delta \frac{dz_2}{dt} &= \Delta \frac{dx_2}{dt} \tan \theta_1 - \frac{1}{\tau}(\Delta x_2 \tan \theta_1 + \Delta b_1 - \Delta z_2)\end{aligned}\right\} \quad . \quad . \quad (42)$$

Third Approximation.

Differentiating equation (40), dividing by $d\,t$, and substituting for $\frac{d^2 x_2}{d t^2}$, $\frac{d^2 y_2}{d t^2}$, $\frac{d^2 z_2}{d t^2}$, their values in equations (14), we have

$$\frac{d^2 r_2}{d t^2} = \frac{1}{r_2}\left(\frac{d x_2^2}{d t^2} + \frac{d y_2^2}{d t^2} + \frac{d z_2^2}{d t^2} - \frac{d r_2^2}{d t^2} - \frac{\mu}{r_2}\right) \quad . \quad . \quad . \quad (43)$$

with this value for $\frac{d^2 r_2}{d t^2}$, find new values for U, W, U', and W' from Eqs. (17); and for a_1, b_1, a_2, b_2, a_3, b_3 from Eqs. (18), by including the terms that were omitted before. Then with the differences between these last values and the next preceding, form equations for the final corrections by writing Δ^2 for Δ in equations (41) and (42). Then the final values of the required quantities become

$$x_2 + \Delta x_2 + \Delta^2 x_2 + \&c. = x,$$
$$y_2 + \Delta y_2 + \Delta^2 y_2 + \&c. = y,$$
$$z_2 + \Delta z_2 + \Delta^2 z_2 + \&c. = z;$$
$$\frac{dx_2}{dt} + \Delta \frac{dx_2}{dt} + \Delta^2 \frac{dx_2}{dt} + \&c. = \frac{dx}{dt} = V_x,$$
$$\frac{dy_2}{dt} + \Delta \frac{dy_2}{dt} + \Delta^2 \frac{dy_2}{dt} + \&c. = \frac{dy}{dt} = V_y,$$
$$\frac{dz_2}{dt} + \Delta \frac{dz_2}{dt} + \Delta^2 \frac{dz_2}{dt} + \&c. = \frac{dz}{dt} = V_z.$$

APPENDIX X.

GEOCENTRIC MOTION.

By the notation of the text, p. 91,

$$a \cos l - \cos L = \rho \,.\, \cos \lambda,$$
$$a \sin l - \sin L = \rho \,.\, \sin \lambda;$$

and by division,

$$\tan \lambda = \frac{a \sin l - \sin L}{a \cos l - \cos L};$$

differentiating,

$$\frac{d\lambda}{\cos^2 \lambda} = \frac{(a \cos l - \cos L)(a \cos l \,.\, dl - \cos L \,.\, dL) + (a \sin l - \sin L)(a \sin l \,.\, dl - \sin L dL)}{(a \cos l - \cos L)^2}$$

$$= \frac{[a^2 - a \cos (L - l)]\, dl + [1 - a \cos (L - l)]\, dL}{(a \cos l - \cos L)^2}.$$

But by Kepler's 3d law,

$$dL : dl :: a^{\frac{3}{2}} : 1;$$

whence

$$dL = a^{\frac{3}{2}} \,.\, dl;$$

which substituted above, and making

$$P = \frac{\cos \lambda}{a \cos l - \cos L},$$

gives

$$d\lambda = P^2 \,.\, [a^2 + a^{\frac{3}{2}} - (a + a^{\frac{5}{2}}) \cos (L - l)] \,.\, dl;$$

and making

$$d\lambda = m, \quad \text{and} \quad dl = n,$$

we have Eq. (124) of the text.

APPENDIX XI.

ON ECLIPSES.

BY MR. W. S. B. WOOLHOUSE, HEAD ASSISTANT ON THE NAUTICAL ALMANAC ESTABLISHMENT.

Eclipses, in all the varieties of aspect which they present to different places on the earth, form an entertaining subject for discussion; and, without considering the public interest generally excited by their prediction and appearance, the use of them, as a test of the degree of perfection of the lunar and solar tables, and in the determination and corroboration of geographical positions, &c., renders their accurate calculation an object of some importance. The popularity of the phenomena naturally called the attention of astronomers, at an early period, into the field of investigation, and several methods of calculation have been adopted by different authors at various periods.

For the general circumstances which take place on the earth, the plan of orthographic projection, though it can only be recognized as affording good approximations, seems to have predominated, and to have been almost exclusively adopted in actual calculations. This method is explained in the astronomical treatises of De la Lande and Delambre, and more recently by Hallaschka, in his *Elementa Eclipsium* (Pragæ, 1816), where an example is to be found at length. Various particulars are laid down in a more accurate manner in *Mémoires sur l'Astronomie Pratique.* Par M. J. Monteiro Da Rocha, traduits du Portugais (Paris, 1808).

The circumstances of an eclipse for a particular place are usually calculated by the "Method of the Nonagesimal," which refers the bodies to the ecliptic, and an example of which may be seen in the work of Hallaschka above mentioned. This part of the subject has also been discussed analytically by Lagrange, in the *Astron. Jahrbuch* for 1782; and Professor Bessel has since made some important additions to the theory, in a paper inserted in the *Astronomische Nachrichten*, vol. vii., No. 151, which is to be found translated in the *Philosophical Magazine*, vol. viii.

As the numerous calculations which may be required for an eclipse, such as of the maps, &c., given in the Nautical Almanac, could not be performed without many perplexing references to different authors, it has been presumed that a complete and systematic set of formulæ would be generally acceptable; and such a conviction has led to the drawing up of the following paper, which contains an extensive classification of useful remarks and formulæ, developed and arranged with a careful view to their practical application, and with the endeavor to establish a direct and uniform mode of conducting each species of calculation.

LIMITS WHICH DETERMINE THE OCCURRENCES OF ECLIPSES.

ELEMENTS.

The following elements, used in the calculation of the limits, have been derived from the tables of Damoiseau, Burckhardt, and Carlini, viz.:

		°	′	″
Moon's horizontal parallax	greatest		61	32
	least		52	50
Sun's horizontal parallax	greatest		0	9
	least		0	8
Moon's semi-diameter	greatest		16	46
	least		14	24
Sun's semi-diameter	greatest		16	18
	least		15	45
Moon's hourly motion in longitude	greatest		38	35
	least		27	47
Sun's hourly motion in longitude	greatest		2	33
	least		2	23
Moon's hourly motion in latitude	greatest		3	4
	least		0	0
Inclination of moon's orbit with ecliptic	greatest	5	20	6
	least	4	57	22

LIMITS.

For the occurrence of an eclipse of the moon:

1. The greatest possible distance of the centres of the moon and earth's shadow at the time of contact, is 63′ 29″.

2. At the time of true ecliptic conjunction of the moon and earth's shadow, or at the time of opposition or full moon, the greatest possible latitude of the moon is 63′ 45″.

3. At the time of opposition, or full moon, the greatest possible distance of the centre of the moon or of the earth's shadow from the ascending or descending node of the moon's orbit is 12° 24′.

For the occurrence of an eclipse of the sun:

1. The greatest possible distance of the centres of the sun and moon, at the time of contact, is 1° 34′ 28″.

2. At the time of true conjunction of the sun and moon, the greatest possible latitude of the moon is 1° 34′ 52″.

3. At the time of true conjunction of the sun and moon, or the time of new moon, the greatest possible distance of the centre of the sun or moon from one of the nodes of the moon's orbit is 18° 36′.

The third of these limits applies to the true place of the node, which may differ considerably from the mean place.

The most convenient and certain limits, however, will be those of the moon's latitude (β), and will be as follows:

1. At the time of full moon an eclipse of the moon will be

$$\left.\begin{array}{l}\text{certain}\\ \text{impossible}\end{array}\right\} \text{when } \beta \left\{\begin{array}{l} < 51'\ 57'' \\ > 63\ \ 45 \end{array}\right.$$

and doubtful between these limits.

For the doubtful cases, an eclipse will result when

$$\beta < \frac{61}{60}(P + \pi - \sigma) + s + 16''$$

in which P, s denote the equatorial horizontal parallax and semi-diameter of the moon, and π, σ those of the sun.

2. At the time of new moon an eclipse of the sun will be

$$\left.\begin{array}{l}\text{certain}\\ \text{impossible}\end{array}\right\} \text{when } \beta \left\{\begin{array}{llll} < 1^\circ & 23' & 15' \\ > 1 & 34 & 52\end{array}\right.$$

and doubtful between these limits.

For the doubtful cases, an eclipse will happen when

$$\beta < (P - \pi) + \sigma + s + 25'$$

PARALLAX.

If a straight line be drawn from the centre of the earth to any assumed place, it will be the radius of the earth for that place, and this radius we shall designate by the letter ρ. This radius ρ, produced upward towards the heavens, will determine what we shall call the *central zenith*, being that point which spherically determines our true position in relation to the centre of the earth. The apparent zenith, however, is naturally determined by a line which is vertical to the observer, and therefore a normal to the spheroidal surface of the earth. The small angular deviation of this normal from the radius of the earth, or the angular distance between the central and apparent zeniths, is what astronomers call "the angle of the vertical;" and, the earth being an oblate spheroid, it is evident that the central zenith will be nearer to the equator than the apparent, and also that the horizontal parallax will always be less than that at the equator, in consequence of the diminution of the earth's radius in proceeding towards the poles. The effect of parallax on the position of a body above the horizon is to augment its zenith distance, and for this we have the well-known relation,

"sin par. in zen. dist. = sin hor. par. × sin app. zen. dist."

This relation will hold strictly for the spheroidal figure of the earth, provided we adopt the central zenith, and that horizontal parallax which appertains to the radius ρ of the place of observation.

Consider the equatorial semi-diameter of the earth as unity, and let γ denote the polar semi-diameter, which, adopting the mean between La Lande and Delambre, will be $\frac{304}{305}$. Let also l be the latitude of the central zenith, or what is usually called the "geocentric latitude," and l' that of the apparent zenith, which may be termed the spheroidal or geographical latitude. Then the co-ordinates of this place, referred, in the plane of its meridian, to the polar axis, will be

$$x = \rho \sin l, \qquad y = \rho \cos l.$$

By the generating ellipse

$$\frac{x^2}{\gamma^2} + y^2 = 1,$$

and therefore for the angle l', which the normal makes with y or the tangent with x, we have

$$\tan l' = -\frac{dy}{dx} = \frac{1}{\gamma^2}\cdot\frac{x}{y} = \frac{\tan l}{\gamma^2},$$

$$\therefore \tan l = \gamma^2 \tan l' \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, the values of x and y, substituted in the above equation of the ellipse, give

$$\rho^2\left(\frac{\sin^2 l}{\gamma^2}+\cos^2 l\right)=1;$$

and hence

$$\rho=\frac{1}{\sqrt{\frac{\sin^2 l}{\gamma^2}+\cos^2 l}}=\frac{1}{\sqrt{1+\frac{1-\gamma^2}{\gamma^2}\sin^2 l}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

To these may be added the following, which are sometimes useful, and directly deducible from the equations (1), (2),

$$x=\rho\sin l=\frac{\gamma^2\tan l'}{\sqrt{1+\gamma^2\tan^2 l'}}=\frac{(1-e^2)\sin l'}{\sqrt{1-e^2\sin^2 l'}} \quad . \; . \; . \; . \quad (3)$$

$$y=\rho\cos l=\frac{1}{\sqrt{1+\gamma^2\tan^2 l'}}=\frac{\cos l'}{\sqrt{1-e^2\sin^2 l'}} \quad . \; . \; . \; . \quad (4)$$

where $e=\sqrt{1-\gamma^2}$ is the eccentricity of the meridian.

Also

$$\rho=\sqrt{x^2+y^2}=\sqrt{\frac{1+\gamma^4\tan^2 l'}{1+\gamma^2\tan^2 l'}}=\sqrt{\frac{\cos l'}{\cos l\cos(l'-l)}} \quad . \; . \; . \; . \quad (5)$$

The equations (1), (2) are convenient, and the latter may be simply resolved by logarithms, thus:

$$\left.\begin{array}{l}\tan\psi=\sin l\sqrt{\dfrac{1-\gamma^2}{\gamma^2}}\\ \rho=\cos\psi\end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

From (1) may also be deduced

$$\left.\begin{array}{l}\tan\chi=\gamma\tan l'=\dfrac{\tan l}{\gamma}=\sqrt{\tan l\tan l'}\\ \tan(l'-l)=\dfrac{1-\gamma^2}{2\gamma}\sin 2\chi\end{array}\right\} \quad . \; . \; . \; . \; . \quad (7)$$

Here we may remark, that in reducing the geographical latitude to the geocentric with the argument l', the auxiliary arc χ, being between the values of l and l', will be a very small quantity in defect of the argument; and that, on the contrary in reducing the geocentric to the geographical latitude, the arc χ will exceed the argument by nearly the same quantity. Therefore, if we assume χ as an argument for the difference $l'-l$, a table formed from the equation

$$\tan(l'-l)=\left(\frac{1-\gamma^2}{2\gamma}\right)\sin 2\chi,$$

or

$$l'-l=\left(\frac{1-\gamma^2}{2\gamma\tan 1''}\right)\sin 2\chi, \text{ in seconds,}$$

will be equally adapted to both reductions, giving nearly the mean between them; and a table so constructed, with the argument χ, signifying either latitude, will answer every necessary degree of accuracy, since the reduction itself is so small. In numbers we have

$$\frac{1-\gamma^2}{2\gamma}=\frac{609}{2\times 304\times 305}, \text{ and its logarithm}=7.51641\ \therefore \log\left(\frac{1-\gamma^2}{2\gamma\tan 1''}\right)=2.83084,$$

and hence

$$l'-l=[2.83084]\sin 2\chi.$$

Thus the following table has been derived:

Difference between the Geographical and Geocentric Latitudes.

Argument: χ, *either Latitude.*

χ		$l'-l$		χ		$l'-l$		χ		$l'-l$	
°	°	′	″	°	°	′	″	°	°	′	″
0	90	0	0	15	75	5	39	30	60	9	47
1	89	0	24	16	74	5	59	31	59	9	58
2	88	0	47	17	73	6	19	32	58	10	9
3	87	1	11	18	72	6	38	33	57	10	19
4	86	1	34	19	71	6	57	34	56	10	28
5	85	1	58	20	70	7	15	35	55	10	37
6	84	2	21	21	69	7	33	36	54	10	44
7	83	2	44	22	68	7	51	37	53	10	51
8	82	3	7	23	67	8	7	38	52	10	57
9	81	3	29	24	66	8	23	39	51	11	3
10	80	3	52	25	65	8	39	40	50	11	7
11	79	4	14	26	64	8	54	41	49	11	11
12	78	4	36	27	63	9	8	42	48	11	14
13	77	4	57	28	62	9	22	43	47	11	16
14	76	5	18	29	61	9	34	44	46	11	17
15	75	5	39	30	60	9	47	45	45	11	17

The difference is to be subtracted from the geographical, or added to the geocentric latitude, whether it be north or south.

It is evident from what has been said, page 334, that if Z denote the true distance of the moon from the central zenith as it would appear at the centre of the earth, and Z' the apparent distance from the same zenith, as seen from the place on the surface, where the radius of the earth is ρ; and furthermore, P the equatorial horizontal parallax, and $z = Z' - Z$, the parallax in altitude, we shall have

$$\sin z = \rho \sin P \sin Z' \qquad (8)$$

Substituting $Z + z$ in the place of Z', and dividing by $\cos z$, we find

$$\tan z = \frac{\rho \sin P \sin Z}{1 - \rho \sin P \cos Z} \qquad (9)$$

which are the usual formulæ for the parallax in altitude.

For the radius ρ of the earth we have $\log \sqrt{\frac{1-\gamma^2}{\gamma^2}} = 8.909435$, and by (6)

$$\tan \psi = [8.909435] \sin l,$$
$$\rho = \cos \psi.$$

The values of ρ so computed are given in the annexed table.

Log. Radius of the Earth.

Argument: Geocentric Latitude.

l	$\log \rho$	l	$\log \rho$	l	$\log \rho$
°		°		°	
0	0·00000	30	9·99964	60	9·99893
1	0·00000	31	9·99962	61	9·99891
2	0·00000	32	9·99960	62	9·99889
3	0·00000	33	9·99958	63	9·99887
4	9·99999	34	9·99955	64	9·99885
5	9·99999	35	9·99953	65	9·99883
6	9·99998	36	9·99951	66	9·99881
7	9·99998	37	9·99948	67	9·99879
8	9·99997	38	9·99946	68	9·99877
9	9·99997	39	9·99943	69	9·99876
10	9·99996	40	9·99941	70	9·99874
11	9·99995	41	9·99938	71	9·99872
12	9·99994	42	9·99936	72	9·99871
13	9·99993	43	9·99934	73	9·99870
14	9·99992	44	9·99931	74	9·99868
15	9·99990	45	9·99929	75	9·99867
16	9·99989	46	9·99926	76	9·99866
17	9·99988	47	9·99924	77	9·99865
18	9·99986	48	9·99921	78	9·99864
19	9·99985	49	9·99919	79	9·99863
20	9·99983	50	9·99916	80	9·99862
21	9·99982	51	9·99914	81	9·99861
22	9·99980	52	9·99911	82	9·99860
23	9·99978	53	9·99909	83	9·99859
24	9·99976	54	9·99907	84	9·99859
25	9·99974	55	9·99904	85	9·99858
26	9·99973	56	9·99902	86	9·99858
27	9·99971	57	9·99900	87	9·99858
28	9·99968	58	9·99897	88	9·99858
29	9·99966	59	9·99895	89	9·99857
30	9·99964	60	9·99893	90	9·99857

PHENOMENA WHICH TAKE PLACE ON THE EARTH GENERALLY.

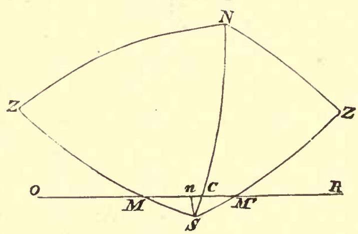
The place on the surface of the earth where the limbs of the sun and moon first appear in contact will be where the penumbra first touches the earth, and, consequently, at this place the apparent contact will be in the horizon, the disk of the moon being wholly above the horizon, and that of the sun below it. The point of contact will be in the same vertical with the two centres; and, therefore, the real as well as the apparent places will be in the same vertical circle; and the lower limb of the moon, being in the horizon, will be depressed by the whole amount of the horizontal parallax which belongs at that time to the latitude of the place. Similarly, the place which first has a central eclipse will be where the straight line through the centres of the sun and moon comes first in contact with the earth, and at this place the centres of both objects will be in the horizon, that of the moon experiencing the whole effect of the horizontal parallax.

The same circumstances will have place where the phenomena finally quit the earth.

Since the apparent places of the sun and moon are so contiguous, and the parallax of the sun so small, it is evident that the relative positions will be the same if we give to the moon the effect of the difference of the parallaxes $P - \pi$, and retain the sun in his true position. This difference $P - \pi$ is therefore the relative parallax, or that which influences the relative position of the bodies. If ρ be the radius of the earth for the place on its surface, the parallax which ought to be used is $\rho\,(P - \pi)$. But in the following investigations, where a place is generally the object of determination, we cannot previously so reduce this relative parallax $P - \pi$. In order therefore to secure the chance of least deviation from the truth in this respect, we shall in these cases reduce the parallax in the first instance to a mean latitude of 45°, so that it will be [9.99929] $(P - \pi)$. We shall consequently, to simplify the analytical expressions, hereafter denote this quantity by the letter P' only; except in one or two instances, where the latitude of the place is known, and where it is always distinctly specified to represent the parallax properly reduced to that latitude, or $\rho\,(P - \pi)$.

I Places where the different Phases are first and last seen on the Earth.

Fig. 5.



Let the whole be referred to the surface of a sphere concentric with the earth; and let $O\,R$ be the relative orbit of the moon, which is generated by the differences of the motions in right ascension and declination, or by the relative motion of the moon; N the north pole; S the sun; $S\,n$ perpendicular to the relative orbit, the nearest approach which we denote by n; C the point where the moon comes in conjunction in right ascension, and $C\,S$ the difference of declination at that time, which we shall denote by contraction, diff. dec. Let also $M\,M'$ be the positions of the moon, when a distance of the centres equal to Δ' first appears on, and finally quits the earth; $M\,S = M'\,S = \Delta$, the corresponding true distance as seen from the centre of the earth; $Z\,Z'$ the zeniths of these places on the earth, which must be respectively in the continuations of $S\,M$, $S\,M'$, in order that the full effect of parallax may be communicated in causing the bodies to approach.

As the apparent zenith distance of the points which experience the greatest effect must be 90°, we may evidently assume $Z\,S = 90°$: for contact of either limb of the moon with the contiguous limb of the sun, we have accurately $Z\,S = (90° - \pi) + \sigma$; for contact of either limb of the moon with the remote limb of the sun $Z\,S = (90° - \pi) - \sigma$; and for contact of the centres $Z\,S = 90° - \pi$. By making $Z\,S = 90°$, the phase will begin with sunrise and end with sunset; and it is evident that no sensible augmentation can affect the semi-diameter of the moon so near the horizon. The true distance $S\,M$ of the centres being Δ, and P' the relative horizontal parallax, the apparent distance Δ' will be $P' \sim \Delta$; and by estimating positive distances from S towards M, in order to have the first occurrence of the phase, it will be $\Delta - P'$;

$$\therefore\ \Delta = P' \pm \Delta'.$$

Here we may notice three limiting aspects,—

(1) When simple or exterior contact of limbs first takes place,

$$\Delta' = s + \sigma, \text{ and } \Delta = P' + s + \sigma.$$

(2) When interior contact of limbs first takes place $\Delta' = s \sim \sigma$; when $s > \sigma$, a total contact first commences with $\Delta' = s - \sigma$; when $s < \sigma$, an annular con tact first commences with $\Delta' = \sigma - s$. Therefore,

If $s > \sigma$, a total eclipse first begins on the earth, when

$$\Delta = P' + s - \sigma.$$

If $s < \sigma$, an annular eclipse first begins on the earth, when

$$\Delta = P' - s + \sigma.$$

(3) When contact of centres first takes place on the earth,

$$\Delta' = 0 \text{ and } \Delta = P'.$$

For the time of true conjunction in right ascension, assume

D, the true declination of the moon;

a, the true difference of right ascension, or ☽'s right ascension *minus* ☉'s right ascension, *in space;*

D_1, the relative motion in declination, or the motion of the moon in declination, *minus* that of the sun, at that time;

a_1, the relative motion in right ascension at the same time;

ι, the inclination of the relative orbit OR with a parallel of declination through the point C, or the angle CSn;

ω, the angle under the distance and the line of nearest approach, or the angle MSn. This angle is always measured on the northern side of the distance, so that when OR falls below S, or when diff. dec. CS is negative, it will exceed 90°.

Then the relations of the figure will give these equations:

$$\tan \iota = \frac{D_1}{a_1 \cos D}; \qquad n = (\text{diff. dec.}) \cos \iota \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{Hourly motion in the orbit} = \frac{D_1}{\sin \iota},$$

$$\text{arc } nC = n \tan \iota.$$

For the time of describing the arc nC, or the interval between the middle of the general eclipse and the time of conjunction, it must be divided by the hourly motion in the orbit. Therefore, t denoting this interval,

$$t = \left(\frac{n \sin \iota}{D_1}\right) \tan \iota.$$

Assume

$$\left.\begin{array}{l} c = 3600'' \times \dfrac{n \sin \iota}{D_1} = [3.55630] \dfrac{n \sin \iota}{D_1} \\ t \textit{ in seconds} = c \tan \iota \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

The sign of will be determined by combining the signs of diff. dec. and D_1; and then

$$\text{time of middle} = \text{time of } ☌ - t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also

$$\cos \omega = \frac{n}{\Delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$Mn = n \tan \omega.$$

Let τ denote the semi-duration of the phase, or the time of describing Mn, and

$$\left.\begin{array}{l} \tau \textit{ in seconds} = c \tan \omega \\ \text{Time of} \left\{\begin{array}{l}\text{beginning}\\ \text{ending}\end{array}\right\} = \text{time of middle} \left\{\begin{array}{c}-\\+\end{array}\right\}\tau \end{array}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

Again, let, at the beginning, the $\angle\ NSZ = a$, and for the ending, the $\angle\ NSZ' = b$; and, these angles being estimated from NS towards the east, we shall have

$$a = (-\iota) - \omega, \qquad b = (-\iota) + \omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and, the sun being supposed in the horizon, $ZS = 90°$, $Z'S = 90°$,

$$\cos NZ = \cos NSZ \sin NS, \qquad \tan ZNS = -\frac{\tan NSZ}{\cos NS},$$

$$\cos NZ' = \cos NSZ' \sin NS, \qquad \tan Z'NS = -\frac{\tan NSZ'}{\cos NS};$$

or

$$\left.\begin{array}{ll} \sin l = \cos a \cos \delta\,; & \tan h = -\dfrac{\tan a}{\sin \delta} \\ \sin l' = \cos b \cos \delta\,; & \tan h' = -\dfrac{\tan b}{\sin \delta} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the latitude and hour angle l, h, relating to the first place, and l', h', to the last. These hour angles are measured from the sun towards the east, so that the longitudes of the places will be determined by subtracting respectively from them the apparent Greenwich times of beginning and ending reduced into degrees and minutes, observing that positive differences will indicate east longitudes and negative differences west longitudes.

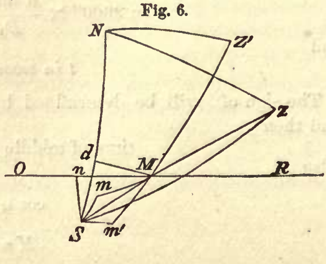
In the preceding formulæ we must use,

$$\begin{array}{c}\text{For beginning and}\\ \text{ending of a}\end{array} \left\{\begin{array}{l}\text{Partial}\\ \text{Total}\\ \text{Annular}\\ \text{Central}\end{array}\right\} \text{Eclipse, } \Delta = \left\{\begin{array}{l} P' + s + \sigma, \\ P' + s - \sigma, \\ P' - s + \sigma, \\ P'. \end{array}\right.$$

II. Rising and Setting Limits.

The places ZZ', thus found, are the two extreme points of a series of places where, at the intermediate times, the same phase will appear in the horizon; and for the phase of external contact of limbs, the curves which these places assume form one of the principal geographical limits of the general eclipse. In the annexed diagram, let M be the place of the moon at a time between the beginning and ending of the partial eclipse. Make $Sm = \Delta'$, $Mm = P'$, and $mZ = 90°$; then at the place Z the moon will appear at m, and have simple external contact with the sun in the horizon. The two triangles SmM, $Sm'M$, will give two such places at each instant, which, on considering the passage of the penumbra over the terrestrial disk, evidently ought to be the

Fig. 6.



case. Since $Mm = P'$ and $Sm = \Delta'$, the possibility of forming the triangles SmM, $Sm'M$, will depend on two conditions for the value of SM, viz., $SM < Mm + Sm$, $SM > Mm - Sm$, or $\Delta < P' + \Delta'$ and $> P' - \Delta'$, that is, Δ must be between the values $P' - \Delta'$ and $P' + \Delta'$: this leads to two species of curves.

1. *When the nearest approach is greater than $P' - \Delta'$.*

Here the formation of the triangles SmM, $Sm'M$, will always be possible during the appearance of the phase on the earth. At the first appearance and final departure of the phase, $SM = Mm + Sm$, the triangle SmM will be simply the line SM, and only one place Z will result. By taking positions of M on both sides of the middle point n, it will also appear that the relative positions of the places ZZ' become inverted, and that the curves described by them must intersect each other at some intermediate place. Hence it appears that the curve of risings and settings commences with a single point, which immediately after divides itself into two points moving in opposite directions on the earth, and which describe two curves intersecting each other, and finally meeting again in a single point, the whole forming one continued curve, returning into itself, and assuming the figure of an 8 much distorted. At the place where they intersect, the phase will begin at sun rise and end at sunset, or it will begin at sunset and end at sunrise.

2. *When the nearest approach is less than $P' - \Delta'$.*

In this case the triangles SmM, $Sm'M$, will resolve into the line SM when $\Delta = P' + \Delta'$ and also when $\Delta = P' - \Delta'$, each of which positions will give only one place Z. Thus it appears that the points Z will form two distinct, oval, and isolated curves, the former curve being generated between the decreasing values $\Delta = P' + \Delta'$ and $\Delta = P' - \Delta'$, and the latter between the increasing values $\Delta = P' - \Delta'$ and $\Delta = P' + \Delta'$. The leading point of the first oval and the terminating point of the second oval are the places where the phase begins and ends on the earth. The terminating point of the first oval and the leading point of the second oval are simply determined by using $\Delta = P' - \Delta'$, and computing the same as for the beginning and ending of a phase on the earth.

Let us now turn our attention to the determination of the two places ZZ', at any time, or for any position of M. Join ZS and draw Md perpendicular to NS.

We shall, throughout our investigation, usually denote Sd by (x), dM by (y), and the $\angle\, dSM$ by S, this angle being estimated from SN towards the east.

To determine these quantities, let the declination of the point $d = (D)$, which will a little exceed that of M, and which is distinguished from it by being placed within a parenthesis; then, supposing NM to be joined, the right-angled spherical triangle NdM will give $\tan (D) = \dfrac{\tan D}{\cos a}$. As a is always small, the difference of the declinations $(D) - D = \tan^{-1} \dfrac{\tan D}{\cos a} - D$ may be arranged in a small table as annexed.

Difference between (D) and D, or a corr.

Arguments: D and a.

D	a = 10′	20′	30′	40′	50′	60′	70′	80′	90′	100′
°	″	″	″	″	″	″	″	″	″	″
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	1	1	1	1	2
2	0	0	0	0	1	1	1	2	2	3
3	0	0	0	1	1	2	2	3	4	5
4	0	0	1	1	2	2	3	4	5	6
5	0	0	1	1	2	3	4	5	6	8
6	0	0	1	1	2	3	4	6	7	9
7	0	0	1	2	3	4	5	7	9	11
8	0	0	1	2	3	4	6	8	10	12
9	0	1	1	2	3	5	7	9	11	13
10	0	1	1	2	4	5	7	10	12	15
11	0	1	1	3	4	6	8	10	13	16
12	0	1	2	3	4	6	9	11	14	18
13	0	1	2	3	5	7	9	12	15	19
14	0	1	2	3	5	7	10	13	17	20
15	0	1	2	3	5	8	11	14	18	22
16	0	1	2	4	6	8	11	15	19	23
17	0	1	2	4	6	9	12	16	20	24
18	0	1	2	4	6	9	13	16	21	26
19	0	1	2	4	7	10	13	17	22	27
20	0	1	3	4	7	10	14	18	23	28
21	0	1	3	5	7	11	14	19	24	29
22	0	1	3	5	8	11	15	19	25	30
23	0	1	3	5	8	11	15	20	25	31
24	0	1	3	5	8	12	16	21	26	32
25	0	1	3	5	8	12	16	21	27	33
26	0	1	3	6	9	12	17	22	28	34
27	0	1	3	6	9	13	17	23	29	35
28	0	1	3	6	9	13	18	23	29	36
29	0	1	3	6	9	13	18	24	30	37

The number of seconds given by this table, which we have denoted by the term a corr., is to be applied so as to increase D, whether it be north or south.

The value of (D) being found by so correcting D with this table, we shall evidently have

$$\left.\begin{array}{c} (x) = (D) - \delta, \qquad (y) = a \cos (D) \\ \tan S = \dfrac{(y)}{(x)}, \\ \Delta = \dfrac{(y)}{\sin S} = \dfrac{(x)}{\cos S}, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (A)$$

the quadrant in which S is to be taken being determined by (x) and (y) as coordinates.

We shall afterwards have frequent occasion to use these quantities.

If t denote the time from the middle of the general eclipse, they may be determined more easily, though less accurately, by means of the following formulæ, which may readily be inferred from what has preceded.

$$\left.\begin{array}{cc} \tan \omega = \dfrac{t}{c}, & \Delta = \dfrac{n}{\cos \omega}, \\ \multicolumn{2}{c}{S = (-\iota) \mp \omega,} \\ (x) = \Delta \cos S, & (y) = \Delta \sin S, \end{array}\right\} \quad \ldots\ldots \quad \text{(B)}$$

the upper sign being for the time t before the middle, and the under sign for the same time after the middle.

Denote the $\angle\, m M S$ by m. In the triangle $m M S$, which may, on account of its smallness, be considered as a plane one, we also have $Mm = P'$, $Sm = \Delta'$. and $SM = \Delta$. Assume

$$\left.\begin{array}{l} p = \dfrac{P' - \Delta'}{2}, \qquad q = \dfrac{P' + \Delta'}{2}, \\ \text{and then} \\ \sin \dfrac{m}{2} = \sqrt{\dfrac{\left(\dfrac{\Delta}{2} - p\right)\left(q - \dfrac{\Delta}{2}\right)}{P' \,.\, \Delta}}, \end{array}\right\} \quad \ldots\ldots \quad (1)$$

As ZS, Zm may be considered as quadrantal arcs, they will be parallel at the extremities S, m; and thus the $\angle\, ZSM = \angle\, mMS = m$. Therefore the $\angle\, NSZ = S \pm m$; and the sun being supposed in the horizon, the spherical triangle NSZ will have $ZS = 90°$, and hence the places Z, Z', will depend on the following formulæ, in which Z is called the place advancing, and Z' the place following.

$$\left.\begin{array}{ll} \textit{Place following,} & \\ \sin l = \cos (S - m) \cos \delta, & \tan h = -\dfrac{\tan (S - m)}{\sin \delta}, \\ \textit{Place advancing,} & \\ \sin l = \cos (S + m) \cos \delta, & \tan h = -\dfrac{\tan (S + m)}{\sin \delta}, \end{array}\right\} \quad \ldots \quad (2)$$

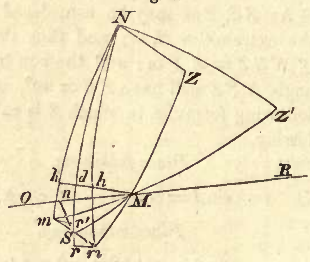
In these expressions the symbol δ represents the declination of the sun at the time for which we calculate; but for common purposes the value of δ at the time of conjunction may be used in all cases.

III. Northern and Southern Limits for any Phase.

The determination of the extreme latitudinal limits of a phase, or of the terrestrial lines whereon that phase will appear as the middle of the local eclipse, is the most complex and unmanageable of all operations which relate to a general eclipse. For any given phase, at different places on the earth, the moon must be so reduced by parallax as to touch a given concentric circle on the solar disk; and if we consider this circle, by way of illustration, to represent, instead of the sun, the disk of the luminous body, the places on the earth which severally see the given phase must be situated in the surface of the penumbral or umbral cone, according as the interfering limb of the moon only approaches or projects over the centre of the sun; that is, the places must all be found in the intersection of this cone with the surface of the earth. This intersection will assume a complete or partial oval

form, according as the cone falls wholly or partially on the earth's illuminated disk. When it falls only partially on the earth, the extreme points will evidently see the sun in the horizon, and be therefore two points belonging to the horizon limits; but in the other case the phase cannot at that instant be seen in the horizon. It is evident then, that these two cases have been already characterized in the discussion of the rising and setting limits. Let us now suppose the bodies to assume consecutive positions, answering to very small intervals of time, the earth also turning round its axis, and we shall have a series of these ovals. It is obvious that the extreme geographical limits of the phase will be represented by curves which envelope all these ovals;—that at each instant the place of limit, by reason of the compound of the motions, will be proceeding relatively in the direction of the tangent to the oval;—that there will be two of these limits when the oval becomes entire during the eclipse, but only one when it is always partial. This is the most popular and natural idea that can be formed of the nature of these limits; and we may here remark, as an inference from what has been said, that if the rising and setting limits of any phase do not extend throughout the general partial eclipse, there will be both a northern and southern limit to that phase; but that, on the contrary, when the rising and setting limits continue throughout the eclipse, there will be only one of these limits to the phase, viz.: a southern limit when the difference of declination at conjunction is positive, and a northern one when that difference is negative.

Fig. 7.



As before, let the system be referred to a sphere concentric with the earth, and let M be the place of the moon; Z, Z', the zeniths of the places which are respectively in the northern and southern limits; and m, m', the corresponding apparent places of the moon. Draw the meridians $N m'$, NS, $N m$, $N Z$, $N Z'$,; also $m\ r$, $m'\ r'$, and $M\ h\ d\ h'$ perpendicular to NS; and assume $S d=(x)$, $d M=(y)$, $m\ h=x$, $h M=y$, $S r=u$, $m\ r=v$, $S m=\triangle'$, $Z m=Z$, $\angle\ N m\ Z=M$, $\angle\ m\ N\ S=a'$, declination of $m=D'$, and the latitude of $Z=l$. Then the $\angle\ m\ N Z=h-a'$, $m\ M=Z'\sin Z$, $x=m\ M\cos M=P'\sin Z\cos M$ and $y=m\ M\sin M=P'\sin Z\sin M$; these by spherics resolve thus:

$$
\begin{aligned}
x &= P'\sin Z\cos M \\
&= P'\left[\sin l\cos D'-\cos l\sin D'\cos(h-a')\right] \\
y &= P'\sin Z\sin M \\
&= P'\cos l\sin(h-a')
\end{aligned}
$$

From these we deduce

$$
\left.
\begin{aligned}
u &= x-(x) \\
&= P'\sin Z\cos M-(x) \\
&= P'\left[\sin l\cos D'-\cos l\sin D'\cos(h-a')\right]-(x) \\
v &= (y)-y \\
&= (y)-P'\sin Z\sin M \\
&= (y)-P'\cos l\sin(h-a')
\end{aligned}
\right\}\ldots\ldots(1)
$$

Let us now keep our attention to the same place Z on the earth, and suppose the system to be in motion as in nature. The hour angle h will increase at the

rate of 15° per hour, and the latitude l will by hypothesis remain unchanged; so that the following equations will ensue:

$$\frac{du}{dt} = -P' \sin 1'' \frac{dD'}{dt} [\sin l \sin D' + \cos l \cos D' \cos (h - a')]$$

$$+ P' \sin 1'' \left(15° - \frac{da'}{dt}\right) \cos l \sin D' \sin (h - a') - \frac{d(x)}{dt}$$

$$= -P' \sin 1'' \frac{dD'}{dt} \cos Z + P' \sin 1'' \left(15° - \frac{da'}{dt}\right) \sin D' \sin Z \sin M - \frac{d(x)}{dt}$$

$$\frac{dv}{dt} = \frac{d(y)}{dt} - P' \sin 1'' \left(15° - \frac{da'}{dt}\right) \cos l \cos (h - a')$$

$$= \frac{d(y)}{dt} - P' \sin 1'' \left(15° - \frac{da'}{dt}\right) (\cos Z \cos D' - \sin Z \sin D' \cos M).$$

Now, in order that m may be the apparent place of the moon at the middle of the eclipse, and consequently her nearest apparent contiguity with the sun, we must have

$\frac{d\triangle'}{dt} = 0$; or since $u^2 + v^2 = \triangle'^2$, $u\frac{du}{dt} + v\frac{dv}{dt} = 0$, which is the condition of limit.

Before we substitute the preceding values of $\frac{du}{dt}, \frac{dv}{dt}$, it may be observed, to avoid complexity, that the quantities $P' \sin 1'' \frac{dD'}{dt}$, $P' \sin 1'' \frac{da'}{dt}$ may be neglected as being very small compared with $P'. 15° \sin 1''$, $\frac{d(x)}{dt}$ and $\frac{d(y)}{dt}$; also that δ may be substituted for D', which will equally serve the purpose of both northern and southern limits. With these modifications we have

$$\left.\begin{aligned} \frac{du}{dt} &= P'. 15° \sin 1'' \sin \delta \sin Z \sin M - \frac{d(x)}{dt} \\ \frac{dv}{dt} &= \frac{d(y)}{dt} - P'. 15° \sin 1'' (\cos Z \cos \delta - \sin Z \sin \delta \cos M) \end{aligned}\right\} \ldots\ldots (2)$$

and, for the condition of limit,

$$u \left[P'. 15° \sin 1'' \sin \delta \sin Z \sin M - \frac{d(x)}{dt}\right]$$
$$+ v \left[\frac{d(y)}{dt} - P'. 15° \sin 1'' (\cos Z \cos \delta - \sin Z \sin \delta \cos M)\right] = 0.$$

Instead of $P' \sin Z \cos M$ put $(x) + u$, and for $P' \sin Z \sin M$ put $(y) - v$, and it becomes

$$u \left[15° \sin 1'' (y) \sin \delta - \frac{d(x)}{dt}\right] + v \left[15° \sin 1'' (x) \sin \delta + \frac{d(y)}{dt}\right]$$
$$- P' v \, 15° \sin 1'' \cos Z \cos \delta = 0;$$

$$\therefore \cos Z = \frac{\frac{u}{P'}\left\{(y) \sin \delta - \frac{\frac{d(x)}{dt}}{15° \sin 1''}\right\} + \frac{v}{P'}\left\{(x) \sin \delta + \frac{\frac{d(y)}{dt}}{15° \sin 1''}\right\}}{v \cos \delta}$$

But, if a_1 denote the true relative motion in right ascension, and D_1 the true relative motion in declination, and D the declination of the moon, at the time of true conjunction,

$$\frac{d(x)}{dt} = D_1; \qquad \frac{d(y)}{dt} = a_1 \cos D;$$

$$\therefore \cos Z = \frac{\frac{u}{P'}\left[(y)\sin\delta - \frac{D_1}{15^\circ \sin 1''}\right] + \frac{v}{P'}\left[(x)\sin\delta + \frac{a_1 \cos D}{15^\circ \sin 1''}\right]}{v \cos\delta}$$

Make now the following assumptions:

$$\left.\begin{aligned}(A) &= \frac{a_1 \cos D}{15^\circ \sin 1''} = [0.58204]\ a_1 \cos D\\ (B) &= \frac{D_1}{15^\circ \sin 1''} = [0.58204]\ D\end{aligned}\right\} \dots\dots\dots\dots (C)$$

$$\left.\begin{aligned}\lambda \sin \nu &= \frac{(B) - (y)\sin\delta}{P' \cos\delta}\\ \lambda \cos \nu &= \frac{(A) + (x)\sin\delta}{P' \cos\delta}\end{aligned}\right\} \dots\dots\dots\dots (3)$$

in which (A), (B) may be used as constant quantities throughout the eclipse, and we get

$$\cos Z = \frac{\lambda}{v}(-u \sin\nu + v \cos\nu).$$

The angle $r\,S\,m$ is equal to the inclination of the apparent relative orbit with the parallel of declination; denote it by ι', and then $u = \Delta' \cos\iota'$, $v = \Delta' \sin\iota'$, and

$$\therefore \cos Z = \lambda \frac{\sin(\iota' - \nu)}{\sin\iota'} \dots\dots\dots\dots\dots\dots (4)$$

which is a concise form of the condition to be fulfilled by Z and ι', in order that the place Z may be situated in the limit of a phase.

Since the $\angle\, M\,S\,d = S$, and the $\angle\, M\,S\,m = 180^\circ - (S + \iota')$, $\angle\, M\,S\,m' = S + \iota'$, we have for the triangle $M\,S\,m$

$$Mm^2 = \Delta^2 + \Delta'^2 \pm 2\,\Delta\,\Delta' \cos(S + \iota').$$

Divide this by P'^2 and we get

$$\sin^2 Z = \frac{\Delta^2 + \Delta'^2}{P'^2} \pm \frac{2\,\Delta\,\Delta'}{P'^2}\cos(S + \iota') \dots\dots\dots (5)$$

for the geometrical relation between S and ι', the upper sign applying to the northern, and the under sign to the southern limit. Add this to the square of the preceding equation (4), and there results

$$\lambda^2 \frac{\sin^2(\iota' - \nu)}{\sin^2\iota'} \pm 2\frac{\Delta\,\Delta'}{P'^2}\cos(S + \iota') + \frac{\Delta^2 + \Delta'^2}{P'^2} = 1 \dots (6)$$

for the determination of the angle ι'.

The solution of this equation is by no means very practicable; but as a small error in the value of Z will not sensibly affect the angle ι', we may have recourse to the following indirect process, in which we first consider the angle ι' to be equal to ι, which in most instances is very nearly so. The letter M designates the the angle $M\,m\,h$.

$$\left.\begin{array}{ll} u = \Delta \cos \iota & D' = \delta \mp u \\ v = \Delta' \sin \iota & a' = \pm \dfrac{v}{\cos D'} \\ \multicolumn{2}{c}{(D) = D + (a - a') \text{ corr.}} \\ y = (a - a') \cos (D) & x = (D) - D' \\ \tan M = \dfrac{y}{x} & \sin Z = \dfrac{x}{P' \cos M} = \dfrac{y}{P' \sin M} \end{array}\right\} \ldots\ldots\ldots (7)$$

the upper signs being for the northern, and the under signs for the southern limit.

Or, if t be the time from the middle of the general eclipse, and ω' the angle under Mm and the line of nearest approach, we shall have

$$Mm \sin \omega' = n \tan \omega = n \frac{t}{c}, \text{ and } Mm \cos \omega' = n \pm \Delta',$$

which, observing that $Mm = P' \sin Z$, give the following equations, wherein E and F are constant for all the computations.

$$\left.\begin{array}{l} \left\{\begin{array}{l} E = \dfrac{n}{c(n \pm \Delta')} \qquad F = \dfrac{n \pm \Delta}{P'} \\ \text{use} \left\{\begin{array}{l}\text{upper}\\ \text{under}\end{array}\right\} \text{sign for} \left\{\begin{array}{l}\text{northern}\\ \text{southern}\end{array}\right\} \text{limit.} \end{array}\right. \\ \left\{\begin{array}{l} \tan \omega' = t.E \qquad \sin Z = \dfrac{F}{\cos \omega'} \qquad M = (-\iota) \mp \omega' \\ \text{use} \left\{\begin{array}{l}\text{upper}\\ \text{under}\end{array}\right\} \text{sign for the interval } t \left\{\begin{array}{l}\text{before}\\ \text{after}\end{array}\right\} \text{the middle.} \end{array}\right. \end{array}\right\} \ldots (8)$$

The sign of the constants E, F, are the same as that of $n \pm \Delta'$; and when this is negative, the angle ω' will be in the second quadrant.

The value of Z determined in this manner will be sufficiently approximate for the purposes of a general map; and where greater minuteness is wanted, it will serve very well to get the angle ι' from the equation (4). For this we have

$$\cot \iota' = \cot \nu - \frac{\cos Z}{\lambda \sin \nu},$$

which may be resolved thus:

$$\sin \phi = \sqrt{\frac{\cos Z}{2 \lambda \cos \nu}} \qquad \tan \iota' = \frac{\tan \nu}{\cos 2\phi} \ldots\ldots\ldots (9)$$

After ι' is so found, which is only wanted roughly, the accuracy of the calculation may be tested by the equation (4); and then we may proceed to a correct computation of MZ, by the equations (7), only using ι' instead of ι. We shall thus have in the spherical triangle ZmN, $ZM = Z$, $Nm = 90° - D'$, and the angle $ZmN = M$; and by spherics the following formulæ:

$$\left.\begin{array}{l} \tan \theta = \tan Z \cos M \\ \tan (h - a') = \dfrac{\sin \theta}{\cos (\theta + D')} \tan M; \qquad \tan l = \tan (\theta + D') \cos (h - a) \\ \text{check} \ \ldots \ \dfrac{\sin \theta}{\cos (\theta + D')} = \dfrac{\sin Z \cos M}{\cos (h - a') \cos l} \end{array}\right\} (10$$

For a map the equations (8) and (10) will alone be amply sufficient. In fact, where a very accurate calculation is wanted, the most satisfactory method will consist in first computing the places roughly; then to reduce the horizontal parallax to the latitude by means of the radius ρ, from the table at page 337, and with

the use of the value of Z, to find the augmented semi-diameter of the moon by means of the table at page 360, and thence the proper value of Δ', and then to follow the equations (3), (9), (4), (7), (10).

The first and last points of these limits will have $Z = 90°$. For these places we have therefore by (5)

$$P'^2 = \Delta^2 + \Delta'^2 \pm 2\,\Delta\,\Delta' \cos(S + \iota').$$

If we assume $\iota' = \iota$, we shall obviously have $S + \iota' = S + \iota = \omega$, and $\Delta \cos(S + \iota') = n$, ω being the angle under the distance Δ and the nearest approach n, as before used.

$$\therefore P'^2 = \Delta^2 + \Delta'^2 \pm 2\,\Delta' n,$$
$$= \Delta^2 - n^2 + (\Delta' \pm n)^2.$$

Consequently

$$\Delta^2 \sin^2 \omega = \Delta^2 - n^2 = P'^2 - (n \pm \Delta')^2,$$

which divided by $\Delta^2 \cos^2 \omega = n^2$, gives

$$\tan \omega = \frac{1}{n}\sqrt{P'^2 - (n \pm \Delta')^2}.$$

Therefore by taking the constant c used in the computation of the beginning and ending of a phase on the earth, we shall have

$$\text{semi-duration} = c \tan \omega = \frac{c}{n}\sqrt{P'^2 - (n \pm \Delta')^2},$$

which may be arranged for calculation as follows:

$$\left.\begin{array}{ll} \cos \omega' = \dfrac{n \pm \Delta'}{P'}, & \text{semi-duration} = c\,\dfrac{P'}{n} \sin \omega', \\ \text{Time of} \begin{Bmatrix} \text{entrance} \\ \text{departure} \end{Bmatrix} = \text{time of middle} \begin{Bmatrix} - \\ + \end{Bmatrix} \text{semi-duration}, & \end{array}\right\} \quad . \quad . \quad (11)$$

The places of entrance and departure of the limits, by continuing the assumption $\iota' = \iota$, may be hence calculated as for the beginning and ending of a phase only using $\delta \mp u$ instead of δ, thus:

$$\left.\begin{array}{ll} \multicolumn{2}{c}{\delta \mp u = D',} \\ a = (-\iota) - \omega, & b = (-\iota) + \omega, \\ \textit{For place of entrance,} & \\ \sin l = \cos a \cos D', & \tan h = -\dfrac{\tan a}{\sin D'}, \\ \textit{For place of departure,} & \\ \sin l = \cos b \cos D', & \tan h = -\dfrac{\tan b}{\sin D'}, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (12$$

Having assumed $\iota' = \iota$, the times and places so computed will only be approximate, though sufficiently near for general purposes. For an accurate calculation, we must first determine the true value of ι'. Since $Z = 90°$, the equations (9) give $\iota' = \nu$, which is also shown by (4). We may, therefore, with the quantities taken out for the respective times of entrance and departure, proceed with the equations (C), (3), use ν instead of ι in (7), and then the final results will be determined by (10). It ought, however, to be observed, that it will be advisable to take the time of entrance in excess to the next higher integral minute, and to reject fractions of a minute in the time of departure; since by fixing on a time a trifle without the actual limits, the value of sin Z would come out greater than

unity, and the calculation rendered useless in consequence. The places so computed will be accurately situated in the limiting lines, and though not strictly the first and last points of these lines, they will be very nearly so.

IV. Determination of the Place where a given Phase will appear both at Sunrise and Sunset.

We have seen (page 341) that when the rising and setting lines of a phase extend throughout the eclipse, they will compose the figure of an 8 much distorted. The point of intersection or nodus is a place where the phase will be seen to begin and end in the horizon; that is, it will either commence at sunrise and end at sunset, or commence at sunset and end at sunrise. At the time of the middle of the eclipse, the sun will therefore be very nearly on the meridian: if diff. dec. and δ are of the same sign, it will be midnight, because the pole of the earth will have the zenith and sun on opposite sides of it; but when those values are of different signs, it will be noon at the place, for then the zenith and sun will be both on the same side of the pole. If τ denote the semi-duration of the eclipse, which begins and ends with the given phase, $\tau \frac{dh}{dt}$ will express the semi-diurnal arc of the sun; and $\therefore -\tan l \tan \delta = \cos\left(\tau \frac{dh}{dt}\right) = \cos\ (\tau\ .\ 15^\circ)$, which being nearly unity, we must have $l \sim \delta$ or Z nearly $= 90^\circ$. Consequently for the values of u, v, $\frac{du}{dt}$, $\frac{dv}{dt}$, at the time of the middle of the eclipse, which will be either noon or midnight, we may assume $\sin\ Z = unity$, and $M = 0^\circ$ or 180°. So we get, from the equations (1) and (2), page 344–5,

$$u = -(x) \pm P', \qquad v = (y),$$

$$\frac{du}{dt} = -\frac{d(x)}{dt}, \qquad \frac{dv}{dt} = \frac{d(y)}{dt} \pm P'\ .\ 15^\circ \sin 1'' \sin \delta.$$

Let μ denote the hourly motion on the apparent relative orbit, and ι' the inclination with a parallel of declination; then

$$\mu \cos \iota' = \frac{dv}{dt},\ \mu \sin \iota' = -\frac{du}{dt};$$

or,

$$\left.\begin{aligned} \mu \sin \iota' &= D_1 \\ \mu \cos \iota' &= a_1 \cos D \pm [9.41796]\, P' \sin \delta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The condition for the greatest phase is $u \frac{du}{dt} + v \frac{dv}{dt} = 0$, or $u \sin \iota' - v \cos \iota' = 0$ that is,

$$[-(x) \pm P'] \sin \iota' - (y) \cos \iota' = 0.$$

If t denote the interval past the time of the true conjunction, we shall have

$$(x) = \text{diff. dec.} + t\, D_1 \text{ and } (y) = t\, a_1 \cos D;$$

$$\therefore [-\text{diff. dec.} \pm P'] \sin \iota' - t\, [D_1 \sin \iota' + a_1 \cos D \cos \iota'] = 0;$$

or, since $D_1 = \left(\frac{D_1}{\sin \iota}\right) \sin \iota$, $a_1 \cos D = \left(\frac{D_1}{\sin \iota}\right) \cos \iota$,

$$[-\text{diff. dec.} \pm P'] \sin \iota' - t \frac{D_1}{\sin \iota} \cos (\iota' \sim \iota) = 0.$$

Assume $$k = \frac{-\text{diff. dec.} \pm P'}{\cos(\iota' \sim \iota)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and then $t = \dfrac{k \sin \iota' \sin \iota}{D_1}$, or since $D_1 = \mu \sin \iota'$,

$$t = \frac{k \sin \iota}{\mu}, \text{ or } t \textit{ in seconds} = [3.55630] \frac{k \sin \iota}{\mu} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When diff. dec. is negative, $M = 180°$, and the lower sign of P' must be used; or, as a general rule, P' must be used with the same sign as that of diff. dec., and, since l nearly $= 90° \sim \delta$, we can previously correct the horizontal parallax for the place by reducing it to a latitude equal to the complement of δ. The value of t being found, we shall have at the place

when diff. dec. and δ have $\left\{\begin{array}{l}\text{the same}\\ \text{different}\end{array}\right\}$ signs, app. time of true ☌ $= \left\{\begin{array}{c}12^h\\ 0^h\end{array}\right\} - t$ (4)

which compared with the Greenwich apparent time of the true conjunction will show the longitude of the place.

For the values of u and v we have

$$u = (-\text{diff. dec.} \pm P') - t D_1 = k \cos(\iota' \sim \iota) - k \sin \iota' \sin \iota = k \cos \iota' \cos \iota,$$
$$v = t\, a_1 \cos D = t D_1 \cot \iota = k \sin \iota' \cos \iota.$$

Let n' be the nearest apparent approach of the centres; and the semi-duration τ will be determined by the equations

$$n' = \frac{v}{\sin \iota'}, \qquad \cos \omega = \frac{n'}{\Delta'}, \qquad \tau = \frac{\Delta' \sin \omega \sin \iota'}{D_1},$$

and thence the latitude by the equation,

$$\tan l = \pm \frac{\cos(\tau \,.\, 15°)}{\tan \delta}.$$

Or, using the above value of v,

$$\cos \omega = \frac{k \cos \iota}{\Delta'}, \qquad \tau = \frac{\Delta' \sin \omega}{\mu}, \qquad \tan l = \pm \frac{\cos(\tau \,.\, 15°)}{\tan \delta} \quad . \quad . \quad (5)$$

the latitude being of the same name as diff. dec.

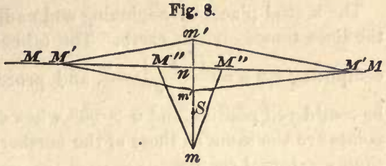
The middle of the eclipse will not have the sun in the horizon, except $k \cos \iota = \Delta$, $\tau = 0$, $l = 90° \sim \delta$, and therefore, unless these particular values should happen, the place will not range exactly in the line whereon the middle of the eclipse is seen at sunrise or sunset; this line, which we are about to notice, will pass the intersection at a higher latitude, and will form a very small triangle with the rising and setting limits.

V Places which will have the Middle of the Eclipse with the Sun in the Horizon.

In the first place, we shall suppose the inclination of the apparent orbit to be the same as that of the true. The condition for the middle of the eclipse will then be simply to have the apparent place of the moon somewhere on the line of nearest approach.

On both sides of S take $S m = S m' = s + \sigma$, and m, m' will be the limits between which the apparent place must be, in order that an eclipse may result. On the orbit make $M' m' = P'$. Then if m' falls between S and n, this will be the first position in which the eclipse can take place. But, if m' falls beyond the

Fig. 8.



point n, the first position of the moon will be at M, where $Mn = P'$; and in this case, for each position between M and M' there will evidently be a position of m' on both sides of the orbit, and consequently two corresponding places on the earth; when the moon arrives at M' the remote point m' will be receding from S, and will at that time get beyond the limit of an eclipse, so that the other point m' only will produce an eclipse under the assigned conditions.

Again, when $m\,n$ is greater than P', it is evident that these limits will continue throughout the whole duration of $M'M'$ or MM. When $m\,n$ is less than P', by making $m\,M'' = P'$ the limits for an eclipse will end at the point M'', and it will be impossible throughout the duration of $M''\,M''$. These two cases are the same as those distinguished in the rising and setting limits, page 340, $s + \sigma$ being the value of $\triangle'$.

To determine the times between which these phases are possible, or the semi-durations answering to the positions M, M', M'', we shall in each instance denote the angle $M\,m\,n$ by the character ω, and the following equations will be readily deduced.

(1) When $n < P' - (s + \sigma)$,

$$\left.\begin{array}{ll} \omega_1 = 90^\circ, & \tau_1 = \dfrac{c\,P'}{n} \\ \cos\omega_2 = \dfrac{n + (s + \sigma)}{P'}, & \tau_2 = \left(\dfrac{c\,P'}{n}\right)\sin\omega_2 \end{array}\right\} \quad \ldots\ldots \quad (1)$$

$\omega_2 > 90^\circ$ when diff. dec. is negative.

These semi-durations will give two times of beginning and ending; the one answering to the point M and the other to the point M''. The middle of an eclipse in the horizon will take place from the first beginning to the second beginning, and from the second ending to the first ending.

The places will be determined by producing $m\,M$ to a distance of 90° from m. If a great circle be drawn through S, so as to be at this point parallel to $m\,M$, it will evidently intersect the former at a distance of 90° and determine the same place. We shall therefore, in supposing the places to be determined in this manner, have the following formulæ:

$$\left.\begin{array}{ll} \text{First place of beginning,} & \omega_1 = 90^\circ, \\ \sin l = -\sin\iota\cos\delta, & \tan h = -\dfrac{\cot\iota}{\sin\delta} \\ \multicolumn{2}{l}{h \text{ must be taken in the 2d semicircle, or between } 0^\circ \text{ and } -180^\circ} \end{array}\right\} \quad \ldots \quad (2)$$

First place of ending,

Change the name of the latitude of the place of beginning, and to the hour angle h apply $\pm 180^\circ$. The results will determine the place of ending.

$$\left.\begin{array}{ll} \text{Second place of beginning,} & \\ a = -\iota - \omega_2, & b = -\iota + \omega_2 \\ \sin l = \cos a\cos\delta, & \tan h = -\dfrac{\tan a}{\sin\delta} \\ \text{Second place of ending,} & \\ \sin l = \cos b\cos\delta, & \tan h = -\dfrac{\tan b}{\sin\delta} \end{array}\right\} \quad \ldots\ldots \quad (3)$$

The second places of beginning and ending will be two of the extreme points of the lines traced on the earth. The other two extremes may be determined by computing $\cos \omega = \frac{(s + \sigma) - n}{P'}$, and proceeding as before, observing that n must be considered positive, and $\omega > 90°$ when diff. dec. is *positive*. These four extreme points are the same as those of the northern and southern limits, the phase being simply external contact.

2) When $n > P' - (s + \sigma)$ and $< s + \sigma$,
The places will be determinable throughout the whole of the first duration found as above. . . . (4)

(3) When $n > s + \sigma$,

$$\cos \omega = \frac{n - (s + \sigma)}{P'}, \qquad \tau = \left(\frac{c P'}{n}\right) \sin \omega,$$

n must here be considered a positive quantity, and ω will be $> 90°$ when diff. dec. is negative.
The phase will continue throughout the whole duration, and the extreme places may be computed from this value of ω according to the equations (3). . . . (5)

Having found the limits between which the phase is possible, the places for any intermediate times may be determined thus,

t denoting the time from the middle,

$$\sin \omega = \left(\frac{n}{c P'}\right) t,$$

$\omega > 90°$ when diff. dec. is *negative*,
and the places by the equations (3).

If $n < s + \sigma$, suppose n to be positive, and compute

$$\cos \omega = \frac{n \sim (s + \sigma)}{P'}; \qquad \tau = \left(\frac{c P'}{n}\right) \sin \omega.$$

Then for times, without the limits of this duration, we may determine four places; two with $\omega < 90°$ and two with $\omega > 90°$, which will all fulfil the necessary conditions.

The preceding results have been derived on the assumption of $\iota' = \iota$. They will be sufficiently approximate for a general drawing of the lines on a map, and more particularly as these phenomena cannot be subject to minute observation. When, however, from local circumstances or otherwise, greater accuracy is wanted, we must use the proper value of ι' and the relative horizontal parallax reduced to the latitude thus determined. Since $Z = 90°$, the condition for the middle of the eclipse, according to the equation (4) page 346, is $\iota' - \nu = 0$ or $\iota' = \nu$. Let the figure at page 344 represent the positions which answer to the particulars of the present case. Then as $Mm = Mm' = P'$, the $\angle Mmm' = \angle Mm'm$. Denote this angle by θ; the angles NmM, $Nm'M$ by M, M'; and we shall have

$$\angle NmS = \nu, \qquad \angle Nm'S = 180° - \nu,$$
$$M = \theta - \nu, \qquad M' = 180° - \nu - \theta,$$
$$\angle MSm = 180° - S - \nu, \qquad \angle MSm' = S + \nu,$$
$$\angle SMm = S + \nu - \theta, \qquad \angle SMm' = 180° - (S + \nu + \theta).$$

With the triangles MSm, MSm', we hence find

$$\sin \theta = \frac{\Delta}{P'} \sin (S + \nu);$$

$$Sm = P' \frac{\sin (S + \nu - \theta)}{\sin (S + \nu)}; \qquad Sm' = P' \frac{\sin (S + \nu + \theta)}{\sin (S + \nu)};$$

which, for computation, may be thus arranged:

$$\left.\begin{array}{ll} g = \dfrac{\sin (S + \nu)}{P'}; & \sin \theta = g \,.\, \Delta; \\ \multicolumn{2}{c}{\theta \text{ to be } + \text{ or } - \text{ but less than } 90^\circ;} \\ Sm = \dfrac{\sin (S + \nu - \theta)}{g}; & M = \theta - \nu; \\ Sm' = \dfrac{\sin (S + \nu + \theta)}{g}; & M' = (180^\circ - \theta) - \nu \end{array}\right\} \quad . \quad . \quad (6)$$

The points m, m', may in some cases be both on the same side of S, and the value of Sm is only necessary to indicate whether any portion of the sun is eclipsed or not. To have an eclipse, Sm, taken as a positive quantity, must be less than $s + \sigma$, and we must only determine a place from the angle M when the corresponding value of Sm is within this limit. If Sm, Sm', taken as positive quantities, are both greater than $s + \sigma$, the middle of an eclipse cannot be seen on the earth under the assumed conditions; on the contrary, if Sm, Sm' so taken are both less than $s + \sigma$, the angles M, M' may both be used, and consequently two places will be determined. In each case, similarly to (3), we adopt the formulæ

$$\left.\sin l = \cos M \cos \delta, \qquad \tan h = -\frac{\tan M}{\sin \delta}\right\} \quad . \quad . \quad . \quad (7)$$

VI. Central Line.

The places which in succession see a central eclipse are evidently determined by producing SM to a distance Z from S, so that

$$\sin Z = \frac{\Delta}{P'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for then the relative parallax P' will bring the centres to a coincidence. To determine the position of the place on the earth for any given time, we have in the triangle NSZ, thus formed, $NS = 90^\circ - \delta$, $\angle NSZ = S$, $SZ = Z$, and hence the following formulæ:

$$\left.\begin{array}{c} \tan \theta = \tan Z \cos S, \\ \theta \text{ to be } + \text{ or } - \text{ and less than } 90^\circ; \\ \tan h = \dfrac{\sin \theta}{\cos (\theta + \delta)} \tan S; \qquad \tan l = \tan (\theta + \delta) \cos h, \\ h \text{ to be in the same semicircle with } S; \\ \text{check} \;.\; . \; \dfrac{\sin \theta}{\cos (\theta + \delta)} = \dfrac{\sin Z \cos S}{\cos h \cos l} \end{array}\right\} \quad . \quad . \quad (2)$$

In the course of the general central eclipse, one of the places on the earth will have the central eclipse at noon. At this instant the bodies will obviously have

true as well as apparent conjunction in right ascension, and $\therefore \Delta =$ diff. dec. and $S = 0$. This place is hence determined thus:

$$\left.\begin{array}{ll} \sin Z = \dfrac{\text{diff. dec.}}{P'}, & l = \delta + Z, \\ \multicolumn{2}{c}{Z \text{ to have the same sign as diff. dec.}} \\ \multicolumn{2}{c}{\text{App. time of true } \sigma = \text{west long. of place,}} \end{array}\right\} \quad \ldots \ldots \quad (3)$$

These equations (1), (2), (3), involve the horizontal parallax P', answering to a mean latitude of 45°, which will be sufficiently near for ordinary purposes. Where an accurate result is wanted, the calculation must be repeated with the use of the equatorial relative parallax properly reduced to the latitude thus determined.

The first and last places on the earth which see a central eclipse, are to be found by the formulæ at pages 338–40.

The preceding discussions comprise all that is necessary for the calculation of the lines which are shown in the maps now inserted in the Nautical Almanac, and which are quite sufficient to indicate the general character of the eclipse that may be expected for any particular place. We might now proceed to show the application of these equations in the resolution of innumerable other curious and interesting problems; but such a field of speculation would not conform with the object of this paper, and may the more willingly be abandoned on the consideration that the means of solution may, in most cases, be readily elicited from the equations already established. The following classification of these equations will be found to exhibit, in a comprehensive form, all that will be requisite to direct and facilitate the operations of the calculator, and relieve the mind from any unnecessary reference or consideration.

Notation.

$D =$ the ☽'s true declination;
$\delta =$ the ☉'s true declination;
$a =$ the true difference of right ascension *in arc*,
 or ☽'s right ascension − ☉'s right ascension;
$D_1 =$ the ☽'s relative motion in declination,
 or ☽'s motion in declination − ☉'s motion in declination,
$a_1 =$ the ☽'s relative motion in right ascension,
 or the motion of the ☽ − that of the ☉;
Diff. dec. = the true difference of declination at ☌ in right ascension,
 viz., ☽'s declination − ☉'s declination, at that time;
$P =$ the ☽'s equatorial horizontal parallax;
$\pi =$ the ☉'s equatorial horizontal parallax;
$P' = [9.99929]\,(P - \pi)$;
$s =$ the ☽'s true semi-diameter;
$\sigma =$ the ☉'s true semi-diameter;
$\Delta =$ the true distance of the centres;
D', a', s', Δ', the apparent values of D, a, s, Δ;
$\omega =$ the angle under Δ and n: in all cases this angle is to be taken positively, and between 0° and 180°.

I.—BEGINNING AND ENDING OF A PHASE ON THE EARTH.

1. (D, D_1 and a_1 at ☌);

$$\tan \iota = \frac{D_1}{a_1 \cos D}; \qquad n = \text{diff. dec.} \times \cos \iota;$$

ι of the same sign as D_1;
n of the same sign as diff. dec.

2. $$c = \frac{n \sin \iota \,[3.55630]}{D_1}; \qquad t = c \tan \iota;$$

sin ι to be found by combining the preceding values of cos ι and tan ι;
sign of t to be determined by diff. dec. $\times D_1$.

3. Time of middle = time of ☌ $- t$;

$$\text{For} \left\{\begin{array}{l}\text{partial}\\ \text{central}\\ \text{total}\\ \text{annular}\end{array}\right\} \text{eclipse, } \Delta = \left\{\begin{array}{l}P' + s + \sigma,\\ P',\\ P' + s - \sigma,\\ P' - s + \sigma;\end{array}\right.$$

$$\cos \omega = \frac{n}{\Delta}; \qquad \tau = c \tan \omega.$$

$$\text{Time of} \left\{\begin{array}{l}\text{beginning}\\ \text{ending}\end{array}\right\} = \text{time of middle} \left\{\begin{array}{c}-\\ +\end{array}\right\} \tau;$$

$$a = (-\iota) - \omega; \qquad b = (-\iota) + \omega.$$

4. Place of beginning, (δ at ☌);

$$\sin l = \cos a \cos \delta; \qquad \tan h = -\frac{\tan a}{\sin \delta};$$

H = apparent Greenwich time of beginning;
longitude east $= h - H$;
h to be in the same semicircle with a.

5. Place of ending, (δ at ☌);

$$\sin l = \cos b \cos \delta; \qquad \tan h = -\frac{\tan b}{\sin \delta};$$

H = apparent Greenwich time of ending;
longitude east $= h - H$;
h to be in the same semicircle with b.

6. For more accurate calculations, reduce the true relative horizontal parallax, by means of the table at p. 337, to the latitudes so determined, and recompute.

II.—RISING AND SETTING LINES.

For partial eclipse, $\Delta' = s + \sigma$.

7. When $n > P' - \Delta'$.
These limits will extend throughout the entire duration of the general eclipse, and form the distorted figure of an 8, the first and last points being the places of beginning and ending on the earth.

8. When $n < P' - \triangle'$.

With $P' - \triangle'$, instead of $\triangle'$, compute as for the times of beginning and ending on the earth; and let these times be t_1, t_2. Then

$$\text{the risings} \left\{ \begin{matrix} \text{begin} \\ \text{end} \end{matrix} \right\} \text{at} \left\{ \begin{matrix} \text{partial beginning,} \\ t_1; \end{matrix} \right.$$

in which interval the first oval will be completed:

$$\text{the settings} \left\{ \begin{matrix} \text{begin} \\ \text{end} \end{matrix} \right\} \text{at} \left\{ \begin{matrix} t_2, \\ \text{partial ending;} \end{matrix} \right.$$

in which interval the second oval will be completed.

The limiting places at the times t_1, t_2, are to be found in the same manner as the places of beginning and ending of a phase on the earth.

9. Places for any times within the limits:

$$\text{Prepare the constants, } p = \frac{P' - \triangle'}{2}, \quad q = \frac{P' + \triangle'}{2},$$

and let t be the time from the middle of the general eclipse;

$$\tan \omega = \frac{t}{c}; \qquad \triangle = \frac{n}{\cos \omega};$$

$$\omega > 90^\circ \text{ when } n \text{ is } -.$$

10. $$S = (-\iota) \mp \omega.$$

$$\text{Use} \left\{ \begin{matrix} \text{upper} \\ \text{under} \end{matrix} \right\} \text{sign for} \left\{ \begin{matrix} \text{before} \\ \text{after} \end{matrix} \right\} \text{the time of middle.}$$

11. $$\sin \frac{m}{2} = \sqrt{\frac{\left(\frac{\triangle}{2} - p\right)\left(q - \frac{\triangle}{2}\right)}{P' \cdot \triangle}},$$

$$\frac{m}{2} \text{ to be less than } 90^\circ \text{ and positive}$$

12. Place following,

$$\sin l = \cos (S - m) \cos \delta; \qquad \tan h = -\frac{\tan (S - m)}{\sin \delta};$$

$$H = \text{apparent Greenwich time;}$$

$$\text{longitude east} = h - H;$$

$$h \text{ to be in the same semicircle with } S - m.$$

13. Place advancing,

$$\sin l = \cos (S + m) \cos \delta; \qquad \tan h = -\frac{\tan (S + m)}{\sin \delta};$$

$$\text{longitude east} = h - H;$$

$$h \text{ to be in the same semicircle with } S + m.$$

14. For a more accurate determination, find the values of D, δ, a for the given time, and $P' = \rho (P - \pi)$ for the latitude; thence

$$(D) = D + (a \text{ corr. from table, p. 342});$$

$$(x) = (D) - \delta; \qquad (y) = a \cos (D);$$

$$\tan S = \frac{(y)}{(x)}; \qquad \triangle = \frac{(y)}{\sin S} = \frac{(x)}{\cos S};$$

$$p = \frac{P' - \Delta'}{2}; \qquad q = \frac{P' + \Delta'}{2}.$$

$$\sin \frac{m}{2} = \sqrt{\frac{\left(\frac{\Delta}{2} - p\right)\left(q - \frac{\Delta}{2}\right)}{P' \cdot \Delta}}.$$

The quadrant of S to be determined by (x), (y), as co-ordinates.
With these values of S, m, compute the places by Nos. 12 and 13.

III. PLACE WHERE THE RISING AND SETTING LIMITS INTERSECT,

When $n > P' - \Delta'$.

15 Find $P' = \rho\,(P - \pi)$. for a latitude equal to the complement of δ at ☌.

$$\mu \sin \iota' = D_1,$$

$$\mu \cos \iota' = a_1 \cos D \pm [9.41796]\, P' \sin \delta,$$

$$k = \frac{-\text{diff. dec.} \pm P'}{\cos(\iota' \sim \iota)}, \qquad t \textit{ in seconds} = [3.55630]\, \frac{k \sin \iota}{\mu}.$$

Use $\left\{ \begin{matrix} \text{upper} \\ \text{under} \end{matrix} \right\}$ signs when diff. dec. is $\left\{ \begin{matrix} \text{positive,} \\ \text{negative.} \end{matrix} \right.$

16. At the place,
When diff. dec. and δ have $\left\{ \begin{matrix} \text{the same} \\ \text{different} \end{matrix} \right\}$ signs, app. time of true ☌ $= \left\{ \begin{matrix} 12^h \\ 0^h \end{matrix} \right\} - t$,
which, compared with the Greenwich apparent time of the true ☌, will determine the longitude of the place.

17.
$$\cos \omega = \frac{k \cos \iota}{\Delta'}, \qquad \tau = \frac{\Delta' \sin \omega}{\mu},$$

$$\tan l = \pm \frac{\cos(\tau \,.\, 15^\circ)}{\tan \delta};$$

l to be of the same name as diff. dec.

IV. PLACES WHERE THE MIDDLE OF THE ECLIPSE IS SEEN WITH THE SUN IN THE HORIZON.

18. When $n < P' - (s + \sigma)$, compute

$$\tau_1 = \frac{c P'}{n}, \qquad \cos \omega_2 = \frac{n \pm (s + \sigma)}{P'}, \qquad \tau_2 = \left(\frac{c P'}{n}\right) \sin \omega_2;$$

using $s + \sigma$ with a sign the same as that of n.

These semi-durations give two times of beginning and ending; the phenomenon will take place on the earth between the times of beginning and between the times of ending.

The places of first and last appearance on the earth to be determied thus:
For first appearance,

$$\sin l = -\sin i \cos \delta, \qquad \tan h = -\frac{\cot \iota}{\sin \delta}.$$

For last appearance, change the name of the latitude of the former place, and to the hour angle h apply $\pm 180°$.

For the extreme points compute also

$$\cos \omega_3 = \frac{n \mp (s + \sigma)}{P'}, \qquad \tau_3 = \left(\frac{c\,P'}{n}\right) \sin \omega_3;$$

using $s + \sigma$ with a sign contrary to that of n.

Then with the values of ω_2, ω_3, proceed as for the beginning and ending of a phase on the earth.

When diff. dec. is $+$, $\left\{\begin{matrix}\omega_2\\ \omega_3\end{matrix}\right\}$ gives points meeting $\left\{\begin{matrix}\text{northern}\\ \text{southern}\end{matrix}\right\}$ limit.

When diff. dec. is $-$, $\left\{\begin{matrix}\omega_2\\ \omega_3\end{matrix}\right\}$ gives points meeting $\left\{\begin{matrix}\text{southern}\\ \text{northern}\end{matrix}\right\}$ limit.

The eclipse will be visible on both sides of the equator.

19. When $n > P' - (s + \sigma)$ and $< s + \sigma$, compute

$$\tau_1 = \frac{c\,P'}{n}.$$

The phenomenon will continue throughout the whole of the duration so found.

The two extreme points will be determined as above with the angle ω_3.

The places of first and last appearance also as above.

20 When $n > s + \sigma$, compute ω_3, τ_3, as above.

The phenomenon will continue throughout the whole duration, and the extreme places will be determined by proceeding with this value of ω as for the beginning and ending of a phase.

These places will in this case be also those of first and last appearance.

21. Places for any time within the limits:

Let t be the time from the middle, and compute

$$\sin \omega = \left(\frac{n}{c\,P'}\right) t.$$

If $n < s + \sigma$, this ω may be taken both greater and less than $90°$ when t is greater than τ_3 before found; and then four places will be determined. In all other cases whatever ω must be $> 90°$ when diff. dec. is negative.

The places to be determined by proceeding with ω as for the beginning and ending of a phase.

22. For a more accurate determination at any time:

Find $P' = \rho\,(P - \pi)$ for the latitude before found.

Find (x), (y), S, and Δ, as in No. 14.

For the time of ☌ form the constants

$$(A) = [0.58204]\, a_1 \cos D, \qquad (B) = [0.58204]\, D_1.$$

Compute ν from the equations,

$$\lambda \cos \nu = \frac{(A) + (x) \sin \delta}{P' \cos \delta}, \qquad \lambda \sin \nu = \frac{(B) - (y) \sin \delta}{P' \cos \delta}.$$

23. Then

$$g = \frac{\sin (S + \nu)}{P'}, \qquad \sin \theta = g \,.\, \Delta,$$

$$\kappa = \frac{\sin (S + \nu - \theta)}{g}, \qquad M = \theta - \nu,$$

$$\kappa' = \frac{\sin (S + \nu + \theta)}{g}, \qquad M' = (180^\circ - \theta) - \nu.$$

θ to be + or — but less than 90°.

24.

$$\sin l = \cos M \cos \delta, \qquad \tan h = -\frac{\tan M}{\sin \delta}.$$

If κ, κ', be both less than $s + \sigma$, the angles M, M', may be both used in these equations, and two places determined. If one of the quantities κ, κ', be greater than $s + \sigma$, the corresponding M will be excluded, and only one place determined with the other value. If κ, κ', be both greater than $s + \sigma$, both computations will be excluded, and the assumed time will be without the limits of the appearance on the earth.

V. NORTHERN AND SOUTHERN LIMITS FOR ANY PHASE.

$$\text{For} \left\{ \begin{array}{l} \text{Partial} \\ \text{Total} \\ \text{Annular} \end{array} \right\} \text{appearance,} \quad \Delta' = \left\{ \begin{array}{l} (s + 6'') + \sigma, \\ (s + 6'') - \sigma, \\ \sigma - (s + 6''). \end{array} \right.$$

6″ is added as a mean augmentation of s.

25. When $n < P' - \Delta'$ both limits will have place.

When $n > P' - \Delta'$ only one limit will have place, viz.:

$$\text{A} \left\{ \begin{array}{l} \text{Northern} \\ \text{Southern} \end{array} \right\} \text{limit when } n \text{ is} \left\{ \begin{array}{l} -, \\ +. \end{array} \right.$$

26. First and last points or places of entrance and departure:

$$\cos \mathrm{w} = \frac{n \pm \Delta'}{P'}, \qquad \tau = \left(\frac{c P'}{n}\right) \sin \mathrm{w};$$

$$\left. \begin{array}{l} \text{upper} \\ \text{under} \end{array} \right\} \text{sign for} \left\{ \begin{array}{l} \text{northern} \\ \text{southern} \end{array} \right\} \text{limit.}$$

$$\text{Time of} \left\{ \begin{array}{l} \text{entrance} \\ \text{departure} \end{array} \right\} = \text{time of middle} \left\{ \begin{array}{l} - \\ + \end{array} \right\} \tau.$$

Places of entrance and departure determined as in Nos. 4 and 5, for the beginning and ending of a phase, using $a = (-\iota) - \mathrm{w}$ and $b = (-\iota) + \mathrm{w}$.

For the appearance of external contact these determinations are included in No. 18, and therefore need not be repeated for these limits.

27. Places for any times within the limits:

Prepare the following constants, using δ at ☌,

$$u = \Delta' \cos \iota, \qquad D' = \delta \mp u, \qquad a' = \pm \frac{\Delta' \sin \iota}{\cos D'},$$

$$E = \frac{n}{c\,(n \pm \Delta')}, \qquad \cos \mathrm{w} = \frac{n \pm \Delta'}{P'} \text{ as above;}$$

$$\left. \begin{array}{l} \text{upper} \\ \text{under} \end{array} \right\} \text{sign for} \left\{ \begin{array}{l} \text{northern} \\ \text{southern} \end{array} \right\} \text{limit.}$$

28. Let t be the time from the middle of the general eclipse,

$$\tan \omega' = t \, . \, E, \qquad \sin Z = \frac{\cos w}{\cos \omega'},$$

$$M = (-\iota) \mp \omega' ;$$

$$\left.\begin{matrix}\text{upper}\\ \text{under}\end{matrix}\right\} \text{sign for} \left\{\begin{matrix}\text{before}\\ \text{after}\end{matrix}\right\} \text{the middle.}$$

29. $$\tan \theta = \tan Z \cos M,$$

$$\tan (h - a') = \frac{\sin \theta}{\cos (\theta + D')} \tan M, \qquad \tan l = \tan (\theta + D') \cos (h - a'),$$

check . . . $$\frac{\sin \theta}{\cos (\theta + D')} = \frac{\sin Z \cos M}{\cos (h - a') \cos l}.$$

$< 90°$, and same sign as $\cos M$; and $h - a'$ to be in the same semicircle with M.

30. For a more accurate determination at any time,

Find $P' = \rho \, (P - \pi)$ for the latitude before found.

Also, with Z find the augmented semi-diameter $s' = s$ + augmentation, from the table annexed.

Augmentation of the ☽'s Semi-diameter.
Argument: *True Zenith Distance Z.*

Z	For P $= 54'$	Var. for 10′ in P.	Z	For P $= 54'$	Var. for 10′ in P.	Z	For P $= 54'$	Var. for 10′ in P.
°	″	″	°	″	″	°	″	″
0	14·0	5·7	30	12·1	4·9	60	6·9	2·9
1	14·0	5·7	31	12·0	4·8	61	6·7	2·8
2	14·0	5·7	32	11·9	4·8	62	6·5	2·7
3	14·0	5·7	33	11·7	4·7	63	6·2	2·6
4	14·0	5·7	34	11·6	4·7	64	6·0	2·5
5	13·9	5·7	35	11·5	4·7	65	5·8	2·4
6	13·9	5·7	36	11·3	4·6	66	5·6	2·3
7	13·9	5·7	37	11·2	4·6	67	5·4	2·2
8	13·8	5·7	38	11·0	4·5	68	5·2	2·1
9	13·8	5·7	39	10·8	4·4	69	4·9	2·0
10	13·8	5·6	40	10·7	4·4	70	4·7	1·9
11	13·7	5·6	41	10·5	4·3	71	4·5	1·8
12	13·7	5·6	42	10·3	4·3	72	4·2	1·7
13	13·6	5·6	43	10·2	4·2	73	4·0	1·6
14	13·6	5·5	44	10·0	4·1	74	3·8	1·5
15	13·5	5·5	45	9·8	4·1	75	3·5	1·4
16	13·4	5·5	46	9·7	4·0	76	3·3	1·3
17	13·4	5·4	47	9·5	3·9	77	3·1	1·2
18	13·3	5·4	48	9·3	3·9	78	2·8	1·1
19	13·2	5·4	49	9·2	3·8	79	2·6	1·1
20	13·1	5·4	50	9·0	3·7	80	2·4	1·0
21	13·0	5·4	51	8·8	3·6	81	2·1	0·9
22	12·9	5·3	52	8·6	3·5	82	1·9	0·8
23	12·8	5·3	53	8·4	3·4	83	1·7	0·7
24	12·7	5·3	54	8·2	3·3	84	1·4	0·6
25	12·6	5·2	55	8·0	3·2	85	1·2	0·5
26	12·5	5·1	56	7·8	3·2	86	1·0	0·4
27	12·4	5·1	57	7·5	3·1	87	0·7	0·3
28	12·3	5·0	58	7·3	3·1	88	0·5	0·2
29	12·2	4·9	59	7·1	3·0	89	0·3	0·1
30	12·1	4·9	60	6·9	2·9	90	0·0	0·0

Then,
$$\text{For}\begin{cases}\text{Partial}\\ \text{Total}\\ \text{Annular}\end{cases}\text{phase, }\Delta'=\begin{cases}s'+\sigma,\\ s'-\sigma,\\ \sigma-s'\end{cases}$$

31. For the time of ☌ form the constants,
$$(A)=[0.58204]\, a_1 \cos D, \qquad (B)=[0.58204]\, D_1.$$

Find the values of D, δ, a, for the given time.
$$(D)=D+(a \text{ corr. from table, page 342}).$$
$$(x)=(D)-\delta, \qquad (y)=a\cos(D),$$
$$\lambda\cos\nu=\frac{(A)+(x)\sin\delta}{P'\cos\delta}, \qquad \lambda\sin\nu=\frac{(B)-(y)\sin\delta}{P'\cos\delta}$$

32. (Z from the first computation),
$$\sin\phi=\sqrt{\frac{\cos Z}{2\lambda\cos\nu}}, \qquad \tan\iota'=\frac{\tan\nu}{\cos 2\phi},$$
$$u=\Delta'\cos\iota', \qquad D'=\delta\mp u,$$
$$v=\Delta'\sin\iota', \qquad a'=\pm\frac{v}{\cos D'},$$
$$(D)=D+(a-a')\text{ corr.}$$
$$y=(a-a')\cos(D), \qquad x=(D)-D',$$
$$\tan M=\frac{y}{x}, \qquad \sin Z=\frac{x}{P'\cos M}=\frac{y}{P'\sin M};$$
$$\begin{Bmatrix}\text{upper}\\ \text{under}\end{Bmatrix}\text{signs for}\begin{Bmatrix}\text{northern}\\ \text{southern}\end{Bmatrix}\text{limit.}$$

Remaining computation the same as in No. 29.

VI. CENTRAL LINE.

33. The computation of the limiting times and places is comprehended under the head, "Beginning and Ending of a Phase on the Earth."

34. Places for any times within the limits:
$$t=\text{the time from the middle.}$$
$$\tan\omega=\frac{t}{c}, \qquad \Delta=\frac{n}{\cos\omega}.$$
$$\omega>90^\circ \text{ when } n \text{ is negative.}$$

35
$$S=(-\iota)\mp\omega;$$
$$\begin{Bmatrix}\text{upper}\\ \text{under}\end{Bmatrix}\text{sign for}\begin{Bmatrix}\text{before}\\ \text{after}\end{Bmatrix}\text{the time of middle.}$$

36 (δ at ☌).
$$\sin Z=\frac{\Delta}{P'}, \qquad \tan\theta=\tan Z\cos S,$$
$$\tan h=\frac{\sin\theta}{\cos(\theta+\delta)}\tan S, \qquad \tan l=\tan(\theta+\delta)\cos h,$$

$$\text{check} \quad . \quad . \quad . \quad \frac{\sin\theta}{\cos(\theta+\delta)} = \frac{\sin Z \cos S}{\cos h \cos l}.$$

θ, same sign as cos S, and less than 90°:
h, same semicircle with S.

37. For a more accurate determination at any time, find P', S, Δ, as in No. 14, and proceed again with these as in No. 36.

38. Place where the eclipse will be central at noon:

(δ at ☌)·

$$\sin Z = \frac{\text{diff. dec.}}{P'}, \qquad l = \delta + Z.$$

Apparent Greenwich time of true ☌ = longitude W.
$Z < 90°$ and same sign as diff. dec.

39. For a more accurate determination, find the horizontal parallax for the latitude, and with it repeat the operation.

[All latitudes in the preceding formulæ are to be recognized as geocentric, and will therefore need reducing by the table at page 336.]

Examples.

For an elucidation of the practical application of the preceding formulæ, we shall take the solar eclipse of May 15, 1836. At the time of new moon, viz. $2^h\ 7^m{\cdot}0$, the moon's latitude β is 25′ 43″, which being less than 1° 23′ 17″, the eclipse is certain. (See the limits at page 333.) The elements of this eclipse, as related to the equator, are

		d. h. m. s.
Greenwich mean time of ☌ in R. A.	May	15 2 21 22·9
☽'s declination	N.	19 25 9·8
☉'s declination	N.	18 57 58·8
☽'s hourly motion in R. A.		30 8·3
☉'s hourly motion in R. A.		2 28·2
☽'s hourly motion in declination	N.	9 58·7
☉'s hourly motion in declination	N.	35·1
☽'s equatorial horizontal parallax		54 23·9
☉'s equatorial horizontal parallax		8·5
☽'s true semi-diameter		14 49·5
☉'s true semi-diameter		15 49·9

from which we prepare the following values:

	° ′ ″		′ ″		
☽'s dec.	+ 19 25 10	☽'s H. M. in R. A.	30 8		
☉'s dec.	+ 18 57 59	☉'s H. M. in R. A.	2 28		
Diff. dec.	+ 27 11	a_1	27 40		
	′ ″		′ ″		
☽'s H. M. in dec.	+ 9 59	☽'s eq. hor. par.	54 24		
☉'s H. M. in dec.	+ 35	☉'s eq. hor. par.	9		
D_1	+ 9 24	Rel. eq. hor. par.	54 15	log.	3·51255
				const.	9·99929
		P'	54 10	log.	3·51184

I. BEGINNING AND ENDING ON THE EARTH.

D_1 + 9′ 24″	. . . .	2·75128	(1)
a_1 27 40	. . . .	3·22011	
		9·53117	
D + 19° 25′·2	cos . .	9·97456	
ι + 19 49	tan . .	9·55661	(2)
	cos . .	9·97349	(3)
diff. dec. + 27′ 11″	. . . .	3·21245	
n + 25 34	. . . .	3·18594	
	sin ι .	9·53010	(2) + (3)
	const. .	3·55630	
		6·27234	(4)
	c . .	3·52106	(4) — (1)
t . . + 19^m 56^s	. c tan ι .	3·07767	
☌ . d. 15 h. 2 21 23			
15 2 1 27	middle of general eclipse		

P' . . . 54′ 10″ = △ for central phase
$s + \sigma$ . . . 30 39
84 49 = △ for partial phase

Partial.			Central.		
n . .		+ 3·18594	n .		+ 3·18594
△ .		3·70663	△ .		3·51184 (log. P')
ω . 72° 27′	cos	+ 9·47931	ω 01° 49′	cos	+ 9·67410
	tan	0·49999		tan	0·27109
	c .	3·52106		c .	3·52106
τ . d. h. m. s. 2 54 57	.	4·02105	τ . d. h. m. s. 1 43 17	.	3·79215
15 2 1 27			15 2 1 27		
14 23 6 30 beginning			15 0 18 10 beginning		
15 4 56 24 ending			15 3 44 44 ending		

	° ′		° ′
$(-\iota)$	— 19 49	$(-\iota)$	— 19 46
ω	72 27	ω	61 49
a	— 92 16	a'	— 81 38
b	+ 52 38	b'	+ 42 0

Place of Partial Beginning.

					h. m. s.
cos a . .	— 8·59715	tan a . .	+ 1·40251	Greenwich time	23 6 30
cos δ . .	+ 9·97576	sin δ . .	+ 9·51191	Equation . .	3 56
sin l . .	— 8·57291	tan h . .	— 1·89060	H in time .	23 10 26
l . . . S.	2° 9′	h . . .	— 89° 16′	H in space .	347° 37′
Reduction	1	H . .	347 37		
Latitude S.	2 10	Longitude W.	76 53		

In the same manner may the places of partial ending and central beginning and ending be calculated, which will come out

		° ′	° ′
Partial ending . .	Long. E.	28 51	Lat. N. 35 13
Central beginning .	Long. W.	98 16	Lat. N. 7 58
Central ending . .	Long. E.	52 41	Lat. N. 44 50

II. RISING AND SETTING LIMITS.

	′ ″		′ ″
P'	54 10		
$s+\sigma=\Delta'$	30 39		
$P'-\Delta'$	23 31	$p=$	11 46
$P'+\Delta'$	84 49	$q=$	42 25

Since $n > P'-\Delta'$, these limits will extend throughout the whole duration of the eclipse; and we may therefore calculate the position of a place for any time between the Greenwich times $14^d\ 23^h\ 6^m\ 30^s$, and $15^d\ 4^h\ 56^m\ 24^s$. As an example, take the time $15^d\ 0^h\ 30^m$.

	d. h. m. s.
Assumed time . . .	15 0 30
Time of middle . . .	15 2 1 27

		t	1 31 27		3·73933
				c	3·52106
	° ′		° ′		
$-\iota$ . . .	− 19 49			tan . . .	0·2182
ω . .	58 51	ω	58 51 .	cos . . .	9·7140
S . .	− 78 40			n	3·18594
m . .	34 2		′ ″		
		Δ	49 24	log Δ . .	3·47191
$S-m$ . .	−112 42	$\frac{1}{2}\Delta$	24 42	comp. . .	6·52809
$S+m$ . .	− 44 38	$\frac{1}{2}\Delta-p$ . .	12 56		2·88986
		$q-\frac{1}{2}\Delta$ . .	17 43		3·02653
				Comp. log P' .	6·48816
					2)18·93264
			° ′		
		$\frac{1}{2}m$ . . .	17 0·9	sin $\frac{1}{2}m$ . . .	9·46632
		m . . .	34 2		

Place Following.

					h. m. s.
cos $(S-m)$	− 9·58648	tan $(S-m)$	+ 0·37850	Greenwich time	0 30 0
cos δ . .	+ 9·97576	sin δ . .	+ 9·51191	Equation . +	3 56
sin l . .	− 9·56224	tan h . .	− 0·86659	H in { time	0 33 56
l . .	S. 21° 24′	h . .	− 82° 15′	H in { space	8° 29
Reduction	8	H . .	8 29		
Latitude .	S. 21 32	Longitude .	W 90 44		

PLACE ADVANCING.

$\cos(S+m)$		$+\ 9\cdot85225$	$\tan(S+m)$		$-\ 9\cdot99444$
$\cos \delta$		$+\ 9\cdot97576$	$\sin \delta$		$+\ 9\cdot51191$
$\sin l$		$+\ 9\cdot82801$	$\tan h$		$+\ 0\cdot48253$
l	N.	42 18	h	$-$	108 13
Reduction		11	H		8 20
Latitude	N	42 29	Longitude	W.	116 42

By taking $S = (-\iota) + \omega$ instead of $(-\iota) - \omega$, similar computations will give the places following and advancing for the interval $t = 1^h\ 31^m\ 27^s$ *after* the time of middle, or for the Greenwich time $15^d\ 3^h\ 32^m\ 54^s$. Much time will be saved by taking the computations two and two in this manner.

III. PLACE WHERE THE RISING AND SETTING LINES INTERSECT.

	90° 0′		
δ	18 58		
l	71 2		
		ρ	$9\cdot99872$
		$P - \pi$	$3\cdot51255$
P'	54′ 5″		$3\cdot51127$
		$\sin \delta$	$+\ 9\cdot51191$
		const.	$9\cdot41796$
	$+\ 4\ 36$		$+\ 2\cdot44114$
		a_1	$3\cdot22011$
		$\cos D$	$9\cdot97456$
	$+\ 26\ 6$		$3\cdot19467$
	$30\ 42 = \mu \cos \iota'$		$+\ 3\cdot26529$
		$\mu \sin \iota'$	$+\ 2\cdot75128$ (D_1)
ι'	17° 1′	$\begin{cases}\tan\\ \cos\end{cases}$	$+\ 9\cdot48599$ $+\ 9\cdot98056$
ι	19 49	μ	$+\ 3\cdot28473$
$\iota' \sim \iota$	2 48		
$-$ diff. dec.	$-\ 27'\ 11''$		
$+\ P'$	$+\ 54\ \ 5$		
	$+\ 26\ 54$		$+\ 3\cdot20790$
		$\cos(\iota' \sim \iota)$	$9\cdot99948$
k	$+\ 3\cdot20842$		$+\ 3\cdot20842$
$\cos \iota$	$+\ 9\cdot97349$	$\sin \iota$	$+\ 9\cdot53010$
	$+\ 3\cdot18191$		$3\cdot55630$
Δ'	$3\cdot26458$		$+\ 6\cdot29482$
$\cos \omega$	$9\cdot91733$	$\log t$	$+\ 3\cdot01009$
$\sin \omega$	$9\cdot75036$		

$\Delta' \sin \omega$.	3·01494			h. m. s.
μ	3·28473	t . . . . . +		0 17 4
				12 0 0
τ	9·73021			
15^s . . .	2·95424	App. time true ☌ .		11 42 56 at the place
				h. m. s.
$8° 4'$ . . .	2·68445			2 21 23
cos . . .	9·99568	Equation		3 56
tan δ . .	9·53615	App. time true ☌ .		2 25 19 at Greenwich
tan l . .	0·45953	Long. in	time	$9^h\ 17^m\ 37^s$ E.
	° ′		space	139° 24′ E.
l . . .	N. 70 51			
Reduction .	7			
Latitude . .	N. 70 58			

Thus we find the required place to be in longitude E. 139° 24′ and latitude N. 70° 58′, where simple contact will have place at sunset and again at sunrise; also the middle of the eclipse would be seen at midnight if it were not intercepted by the opacity of the earth. The duration of the eclipse will correspond with the duration of the night, and therefore no portion of it will be visible.

IV. PLACES WHERE THE MIDDLE OF THE ECLIPSE HAS THE SUN IN THE HORIZON.

In the present case n is $> P' - (s + \sigma)$ and $< s + \sigma$. We must therefore proceed as in No. 19.

1. *For the extreme points,*

		c . . .	3·52106
		P' . .	3·51184
			7·03290
n . .	+ 25′ 34″	. . . .	3·18594
$-(s+\sigma)$	− 30 39	$\frac{cP'}{n}$ . .	3·84696 (1)
	− 5 5	. . . .	− 2·48430
		P' . .	3·51184
	° ′	cos .	− 8·97246
ω_3 . . .	95 23	. . . sin . .	9·99808 (2)
	° ′	h. m. s.	
$(-\iota)-$	19 49	τ_3 56 39 . . .	3·84504 (1) + (2)
ω_3	95 23	2 1 27 time of middle	
a −	115 12	0 4 48 time of beginning	
b +	5 34	3 58 6 time of ending	

PLACE OF BEGINNING, OR FIRST EXTREME PLACE.

					h. m. s.
cos a . .	$-9\cdot62918$	tan a .	$+0\cdot32738$	Greenwich time	0 4 48
cos δ . .	$+9\cdot97576$	sin δ . .	$+9\cdot51191$	Equation . .	3 56
sin l . .	$-9\cdot60494$	tan h .	$-0\cdot81547$	H in time	0 8 44
	° ′		° ′	H in space	2° 11′
l . .	S. 23 45	h .	-81 18		
Reduction	8	H .	$+$ 2 11		
Latitude	S. 23 53	Longitude	W. 83 29		

PLACE OF ENDING, OR LAST EXTREME PLACE.

					h. m. s.
cos b . .	$+9\cdot39664$	tan b . .	$+0\cdot58943$	Greenwich time	3 58 6
cos δ . .	$+9\cdot97576$	sin δ . .	$+9\cdot51191$	Equation . .	3 56
sin l . .	$+9\cdot37240$	tan h .	$-1\cdot07752$	H in time	4 2 2
	°		° ′	H in space	60° 31′
l . .	N. 13 38	h .	$+94$ 47		
Reduction	5	H .	60 31		
Latitude	N. 13 43	Longitude	E. 34 16		

2. *For the extreme times,*

the value of τ_1 taken out from the preceding logarithm of $\frac{cP'}{n}$ is $1^h\ 57^m\ 10^s$

h. m. s.	
2 1 27	time of middle
1 57 10	. . τ_1
0 4 17	first appearance
3 58 37	last appearance

PLACE OF FIRST APPEARANCE.

					h. m. s.
sin ι . .	$+9\cdot53010$	cot ι . .	$+0\cdot44339$	Greenwich time	0 4 17
cos δ . .	$+9\cdot97576$	sin δ . .	$+9\cdot51191$	Equation .	3 56
sin l . .	$-9\cdot50586$	tan h . .	$-0\cdot93148$	H in time	0 8 13
	° ′		° ′	H in space	2° 3′
l . .	S. 18 42	h . .	-83 19		
Reduction	7	H .	$+$ 2 3		
Latitude	S. 18 49	Longitude	W. 85 22		

PLACE OF LAST APPEARANCE.

					h. m. s.
	° ′		° ′		
Latitude	N. 18 49		$-$ 83 19	Greenwich time	3 58 37
			180 0	Equation . .	3 56
		h .	$+$ 96 41	H in time	4 2 33
		H .	$+$ 60 38	H in space	60° 38
		Longitude	E. 36 3		

For the computation of places in this line, we have therefore the whole range between the Greenwich mean times $0^h\ 4^m\ 17^s$ and $3^h\ 58^m\ 37^s$. As an example, take the time $1^h\ 30^m$.

	h. m. s.		
Time of middle	2 1 27		
	1 30		
t . . .	0 31 27	. . . .	3·27577
	° ′	$\frac{cP'}{n}$.	3·84696
$(-\iota)$ . .	− 19 49		
ω . . .	15 34	. sin .	9·42881
a . . .	− 35 23		
b . . .	− 4 15		

cos a . .	+ 9·91132	tan a . .	− 9·85140	Greenwich time	h. m. s. 1 30 0
cos δ . .	+ 9·97576	sin δ . .	+ 9·51191	Equation .	3 56
sin l . .	+ 9·88708	tan h . .	+ 0·33949	H in time	1 33 56
	° ′		° ′	H in space	23° 29′
l . .	N. 50 27	h . .	− 114 35		
Reduction	11	H .	+ 23 29		
Latitude	N. 50 38	Longitude	W. 138 4		

By similarly using the angle b, we shall find the position for the interval $31^m\ 27^s$ after the time of middle, or for the time $2^h\ 32^m\ 54^s$; thus,

cos b . .	+ 9·99880	tan b . .	− 8·87106	Greenwich time	h. m. s. 2 32 54
cos δ . .	+ 9·97576	sin δ . .	+ 9·51191	Equation	3 56
sin l . .	+ 9·97456	tan h . .	+ 9·35915	H in time	2 36 50
	° ′		° ′	H in space	39° 13′
l . .	N. 70 35	h . .	− 167 7		
Reduction	7	H .	+ 39 13		
Latitude	N. 70 42	Longitude	W. 206 20 / E. 153 40		

The places may be computed by two together in this way; and it will perhaps be a little more convenient to assume a value of t in the first instance. We may take any value which does not exceed τ_1 or $1^h\ 57^m\ 10^s$. In the present example we should take $t = 31^m\ 27^s$, and begin as under:

	° ′	log t . .	3·27577		h. m. s.
$(-\iota)$ . .	− 19 49			Time of middle .	2 1 27
ω . .	15 34	$\frac{cP'}{n}$	3·84696	t . . .	31 27
a . .	− 35 23	sin ω . .	9·42881	Time before middle	1 30 0
b . .	− 4 15			Time after middle	2 32 54

and then proceed for the places as above.

V. NORTHERN AND SOUTHERN LIMITS.

1. For the partial phase, *we have only southern line of simple contact.*

Constants E, cos w, D', a'.

$s + 6''$ . .	14′ 56″					
σ	15 50					
Δ' . . .	30 46					
n . . .	+ 25 34	. . .	+ 3·18594			
$n - \Delta'$	— 5 12	. . .	— 2·49415	. . . .		— 2·49415
			— 0·69179			
		c . .	3·52106	P' . .		3·51184
		E . .	— 7·17073	cos w .		— 8·98231
		Δ' .	3·26623	. . . .		3·26623
		cos ι .	+ 9·97349	sin ι .		+ 9·53010
		log u .	+ 3·23972			+ 2·79633
		u .	28′ 57″	cos D' .		+ 9·97448
		δ .	+ 18° 57 59	log a' .		— 2·82185
		D'	+ 19 26 56	a' .		— 11′ 4″

The extreme places will be the same as those which have the middle of the eclipse with the sun in the horizon, page 366; and we may compute for any time between the corresponding times of beginning and ending, viz.: $0^h\ 4^m\ 48^s$ and $3^h\ 58^m\ 6^s$; or we may take any value of t less than $1^h\ 56^m\ 39^s$. For an example, take $t = 0^h\ 58^m\ 33^s$.

	h. m. s.			° ′		
					t .	3·54568
Time of middle .	2 1 27	$(-\iota)$ . .	—	19 49	E .	— 7·17073
t . .	0 58 33	ω' .		100 53	tan ω' .	— 0·71641
Before middle .	1 2 54	. . M .	—	120 42	cos ω'	— 9·27571
After middle .	3 0 0	. . M .	+	81 4	cos w .	— 8·98231
		Z .	+	30° 35′·3	sin Z .	+ 9·70660
					tan Z .	+ 9·77167

Remaining calculation for the time $3^h\ 0^m\ 0^s$.

	tan Z . .	+ 9·77167	sin Z . .	+ 9·70660
	cos M . .	+ 9·19113	cos M .	+ 9·19113
θ + 5° 14·7	tan θ . .	+ 8·96280		+ 8·89773
D' + 19 26·9	sin θ . .	+ 8·96098	Comp. cos $(h - a')$	+ 0·07456
$\theta + D'$ + 24 41·6	cos . . .	+ 9·95835	Comp. cos l . .	+ 0·03035
		+ 9·00263	. check . . .	+ 9·00264
	tan M . .	+ 0·80357		
$h - a'$ + 32° 37′·2 .	tan . . .	+ 9·80620		
a' — 11·1	cos . . .	+ 9·92544		h. m. s.
h . . 32 26·1	tan $(\theta + D')$	+ 9·66258	Greenwich time	3 0 0
			Equation . .	3 56
	tan l . .	+ 9·58802	H in time .	3 3 56
	l . .	N. 21° 10′·2	H in space	+ 45° 59′
	Reduction .	7·6	h . . .	+ 32 26
	Latitude .	N. 21 18·–	Longitude .	W. 13 33

The calculation for the time $1^h\ 2^m\ 54^s$ is to be performed in this manner, with the same values of tan Z, sin Z, only taking the value of $M = -120°\ 42'$.

A more accurate Calculation for the Time $3^h\ 0^m\ 0^s$.

Constants (A), (B).

a_1	+ 3·22011		
cos D . . .	9·97456	D_1	+ 2·75128
const. . . .	0·58204	. . . . . .	0·58204
	3·77671		3·33332
(A)	+ 1° 39′ 40″	(B)	+ 0° 35′ 54″

These constants may serve for the computations at all times. For the present example the following is the process employed:

	° ′ ″		′ ″		
D .	+ 19 31 34 (D)	a . .	+ 17 49·0		
a corr. .	1				
δ . . .	+ 18 58 21	a . .	+ 3·02898	$P - \pi$ . . .	3·51255
(x) . . .	+ 0 33 14	cos (D)	+ 9·97428	ρ . .	9·99982
log (x) .	+ 3·29973	log (y)	+ 3·00326	P' . .	3·51237
sin δ	+ 9·51204		+ 9·51204	cos δ . .	9·97574
	+ 2·81177		+ 2·51530	P' cos δ .	3·48811
(x) sin δ	+ 0° 10′ 48″	(y) sin δ	+ 0° 5′ 28″	cos Z . .	9·93493
(A) . .	+ 1 39 40	(B) .	+ 0 35 54	2 λ cos ν .	0·63430
	+ 1 50 28		+ 0 30 26	$\sin^2 \phi$ . .	9·30063
log .	+ 3·82138	log .	+ 3·26150	sin ϕ . .	9·65032
P' cos δ	3·48811		3·48811	ϕ	26° 33′·1
λ cos ν .	+ 0·33327	λ sin ν .	+ 9·77339	2 ϕ . . .	53 6·2
2 . .	0·30103	λ cos ν .	+ 0·33327	cos 2 ϕ . .	9·77843
2 λ cos ν	+ 0·63430	tan ν .	+ 9·44012		+ 9·44012
				tan ι' . .	+ 9·66169
		ι . . .	+ 24° 38′·9	cos ι . .	+ 9·95851
				sin ι' . .	+ 9·62020

	′ ″				
s . . .	14 50				
aug. . .	12				
s'	15 2				
σ	15 50				
Δ'	30 52		3·26764		3·26764
		cos ι' . .	+ 9·95851	sin ι' . .	+ 9·62020
			+ 3·22615		2·88784
		u	+ 0° 28′ 3″	cos D'	9·97451
		δ	+ 18 58 21		2·91333
		D' . . .	+ 19 26 24		

		a' . . .	$-0°\ 13'\ 39''$
		a . . .	$+0\ 17\ 49$
D . . .	$+19°\ 31'\ 34''$	$a-a'$ . .	$+\ 31\ 28$
$(a-a')$ corr.	3	log .	$+3·27600$
(D) . .	$+19\ 31\ 37$	cos . .	$+9·97428$
D' . . .	$+19\ 26\ 24$	y . . .	$+3·25028$
x . . .	$+0\ 5\ 13$	. . . .	$+2·49554$
M . . .	$+80°\ 1'\ 4$	tan . .	$+0·75474$
		sin . .	$+9·99338$
		P' . .	$+3·51237$
			$+3·50575$
Z . . .	$+33°\ 43'·9$	sin . .	$+9·74453$
		cos . .	$+9·91994$

	tan Z . .	$+9·82459$	sin Z . . .	$+9·74453$
	cos M . .	$+9·23864$	. . . . . .	$+9·23864$
θ . . $+\ 6°\ 35'·9$	tan θ . .	$+9·06323$		$+8·98317$
D' . $+19\ 26·4$	sin θ . .	$+9·06034$	comp. cos $(h-a')$	$+0·09214$
$\theta+D'$ $+26\ 2·3$	cos . . .	$+9·95352$	comp. cos l .	$+0·03151$
		$+9·10682$	check . .	$+9·10682$
	tan M . .	$+0·75474$		
$h-a'$ $+36°\ 1'·1$	tan . . .	$+9·86156$		h. m. s.
a' $-\ 13·7$	cos . . .	$+9·90786$	Greenwich time	3 0 0
h . $+35\ 47·4$	tan $(\theta+D')$	$+9·68892$	Equation . .	3 56
	tan l . .	$+9·59678$	H in time .	$+\ 3\ 3\ 56$
	l	N. $21°\ 33'·7$	H in space	$+45°\ 59'·0$
	Reduction .	$7·7$	h	$+35\ 47·4$
	Latitude .	N. $21\ 41·4$	Longitude .	W. $10\ 11·6$

This result differs materially from the former one; but we are not to infer that the former position is so far wide of the truth. In general the second determination may be considered as an almost accurate point in the limit, and though the first result be some distance apart, yet it will always be very near to the limiting *line*, sufficiently near indeed for the mapping of the lines. By direct calculations of the eclipse for these places, the former will have an eclipse of about $\frac{1}{200}$ of the sun's diameter, and the latter about $\frac{1}{10000}$ of the diameter, which is too small to be perceptible.

2. FOR THE ANNULAR PHASE, *we have both northern and southern limits.*

Constants E, cos w, D', a', for northern limit.

$s+6''$ . .	$14'\ 56''$			
σ . . .	$15\ 50$			
Δ . .	$0\ 54$			
n	$+25\ 34$	$+3·18594$		
$n+\Delta'$ . .	$+26\ 28$	$+3·20085$	. . . .	$+3·20085$
		$+9·98509$		
	c . .	$3·52106$	P' . .	$+3·51184$
	E . .	$+6\ 46403$	cos w .	$+9·68901$

			w . .	+ 60° 44′·8
			sin w .	+ 9·94075
			$\frac{cP'}{n}$ . .	+ 3·84696
		h. m. s.		
τ . .		1 42 14	. . . .	+ 3·78771
Δ' .		1·73239	. . . .	1·73239
$\cos \iota$.	+	9·97349	$\sin \iota$.	+ 9·53010
$\log u$.	+	1·70588		+ 1·26249
u . .	+	0° 0′ 51″	$\cos D'$.	+ 9·97579
δ . .	+	18 57 59		+ 1·28670
D' . .	+	18 57 8	a' . .	+ 0° 0′ 19″

The semi-duration of the northern limit on the earth is therefore $1^h\ 42^m\ 14^s$, and we may calculate for any value of t not exceeding this. A calculation of the extreme places on the earth is to be performed the same as for the beginning and ending of a phase on the earth, and will be unnecessary here. As an example, for a time within the limits, we shall take $t = 1^h\ 10^m\ 0^s$.

	h. m. s.		° ′		
				t .	3·62325
Time of middle .	2 1 27	$(-\iota)$ . .	— 19 49	E .	+ 6·46403
t . . .	1 10 0	ω' . .	50 43	$\tan \omega'$.	+ 0·08728
Before middle .	0 51 27	. . M . .	— 70 32	$\cos \omega'$.	+ 9·80147
After middle .	3 11 27	. . M . .	+ 30 54	$\cos$ w .	+ 9·68901
		Z . .	+ 50 31·3	$\sin Z$.	+ 9·88754
				$\tan Z$.	+ 0·08423

Remaining calculation for the time $3^h\ 11^m\ 27^s$.

	$\tan Z$ . .	+ 0·08423	$\sin Z$. . . .	+ 9·88754
° ′	$\cos M$ . .	+ 9·93352	$\cos M$ . . .	+ 9·93352
θ + 46 10·2	$\tan \theta$ . .	+ 0·01775		+ 9·82106
D' + 18 57·1	$\sin \theta$ . .	+ 9·85817	comp. cos $(h - a')$	+ 0·15622
$\theta + D'$ + 65 7·3 .	cos . . .	+ 9·62397	comp. cos l . .	+ 0·25693
		+ 0·23420	. check . . .	+ 0·23421
	$\tan M$ . .	+ 9·77706		
° ′	tan . . .	+ 0·01126		h. m. s.
$h - a'$ + 45 44·6 .			Greenwich time	3 11 27
a' . + 0·3	cos . . .	+ 9·84378	Equation . .	3 56
h . + 45 44·9	$\tan (\theta + D')$	+ 0·33374		
	$\tan l$ . .	+ 0·17752	H in time .	3 15 23
	l . .	N. 56° 23′·8	H in space	+ 48° 51′
	Reduction .	10 ·4	h	+ 45 45
	Latitude .	N. 56 34 ·—	Longitude .	W. 3 6

The calculation for $0^h\ 51^m\ 27^s$ is to be performed in the same manner, with $M = -70°\ 32'$.

A MORE ACCURATE CALCULATION FOR THE TIME $3^h\ 11^m\ 27^s$.

	° ′ ″		′ ″		
D . .	+ 19 33 27 } (D)	α . .	+ 23 6		
α corr. .	1 }				
δ . .	+ 18 58 28	α . .	+ 3·14176	$P - \pi$ . .	3·51255
(x) . .	+ 0 35 0	cos (D)	+ 9·97419	ρ . . .	9·99901
log (x) .	+ 3·32222	log (y)	+ 3·11595	P' . . .	3·51156
sin δ .	+ 9·51208		+ 9·51208	cos δ . .	9·97574
	+ 2·83430		+ 2·62803	P' cos δ .	3·48730
(x) sin δ	+ 0° 11′ 23″	(y) sin δ	+ 0° 7′ 5″	cos Z . .	9·80331
(A) . .	+ 1 39 40	(B) .	+ 0 35 54	2 λ cos ν .	0·63740
{	+ 1 51 3	{	+ 0 28 49	$\sin^2 \phi$ . .	9·16591
{ log .	+ 3·82367	{ log .	+ 3·23779	sin ϕ . .	9·58296
P' cos δ	3·48730		3·48730	ϕ . . .	22° 30′·4
λ cos ν .	+ 0·33637	λ sin ν .	+ 9·75049	2 ϕ . . .	45 0·8
2 . .	0·30103	λ cos ν .	+ 0·33637	cos 2 ϕ . .	9·84939
2 λ cos ν	+ 0·63740	tan ν .	+ 9·41412		+ 9·41412
				{ tan ι' . .	+ 9·56473
		ι' . .	+ 20° 9′·4 .	{ cos ι' . .	+ 9·97255
				{ sin ι' . .	+ 9·53728
	′ ″				
s . . .	14 50				
aug. . . .	9				
s' . . .	14 59				
σ . .	15 50				
Δ . .	0 51		1·70757		1·70757
		cos ι' . .	+ 9·97255	sin ι' . .	+ 9·53728
			+ 1·68012		+ 1·24485
		u . . .	+ 0° 0′ 48″	cos D' .	+ 9·97577
		δ . . .	+ 18 58 28		+ 1·26908
		D' . . .	+ 18 57 40	α' . . .	+ 0° 0′ 19″
				α . . .	+ 0 23 6
		D . . .	+ 19 33 27	{ $\alpha - \alpha'$ . .	+ 0 22 47
		$(\alpha - \alpha')$ corr.	1	{ log . .	+ 3·13577
		(D) . .	+ 19 33 28	cos . .	+ 9·97419
		D' . . .	+ 18 57 40	y . . .	+ 3·10996
		x . . .	+ 0 35 48		+ 3·33203
				{ tan . .	+ 9·77793
		M . . .	+ 30° 57′·1 .	{ cos . .	+ 9·93328
					+ 3·39875
				P' . .	+ 3·51156
				{ sin . .	+ 9·88719
		Z . . .	+ 50° 27′·9	{ cos . .	+ 9·80383

		tan Z . .	+ 0·08336	sin Z . . .	+ 9·88719
	° ′	cos M . .	+ 9·93328	cos M . . .	+ 9·93328
θ . .	+ 46 5·8	tan θ . .	+ 0·01664		+ 9·82047
D' .	+ 18 57·7	sin θ . .	+ 9·85764	comp. cos $(h - a')$	+ 0·15588
$\theta + D'$	+ 65 3·5 .	cos . . .	+ 9·62499	comp. cos l .	+ 0·25630
			+ 0·23265 .	check . . .	+ 0·23265
		tan M . .	+ 9·77793		
	° ′				
$h - a'$	+ 45 41·9 .	tan . . .	+ 0·01058		h. m. s.
a' .	+ 0·4	cos . . .	+ 9·84412	Greenwich time	3 11 27
h .	+ 45 42·3	tan $(\theta + D')$	+ 0·33249	Equation . .	3 56
		tan l . .	+ 0·17661	H in time .	+ 3 15 23
		l . . .	N. 56° 20′·5	H in space	+ 48° 50′·8
		Reduction .	10 ·4	h	+ 45 42 ·3
		Latitude .	N. 56 30 ·9	Longitude .	W. 3 8 ·5

VI. CENTRAL LINE.

We have, at page 363, found the semi-duration of the central appearance on the earth to be 1ʰ 43ᵐ 17ˢ, which is therefore the greatest value of t for this phase. As an example for a time within the limits, take the same value of t as in the two preceding examples.

	h. m. s.		° ′	t . .	3·62325
Time of middle .	2 1 27	$(-\iota)$ . .	— 19 49	c . .	3·52106
t . .	1 10 0	ω . .	+ 51 41	tan ω . .	0·10219
Before middle .	0 51 27 . .	S . .	— 71 30	cos ω . .	9·79246
After middle .	3 11 27 . .	S . .	+ 31 52	n . .	3·18594
				Δ . .	3·39348
				P . .	3·51184
		Z . .	+ 49 35·6	sin Z . .	9·88164
				tan Z . .	0·06994

Remaining computation for the time 3ʰ 11ᵐ 27ˢ.

		tan Z . .	+ 0·06994	sin Z	+ 9·88164
	° ′	cos S . .	+ 9·92905	cos S . . .	+ 9·92905
θ	+ 44 56·0	tan θ . .	+ 9·99899		+ 9·81069
δ	+ 18 58·0	sin θ . .	+ 9·84898	comp. cos h . .	+ 0·15009
$\theta + \delta$	+ 63 54·0 .	cos . . .	+ 9·64339	comp. cos l . .	+ 0·24480
			+ 0·20559 .	check . . .	+ 0·20558
		tan S . .	+ 9·79354		
	° ′				
h .	+ 44 56·6 .	tan h . .	+ 9·99913		h. m. s.
		cos h . .	+ 9·84991	Greenwich time	3 11 27
		tan $(\theta + \delta)$.	+ 0·30990	Equation . .	3 56
		tan l . .	+ 0·15981	H in time .	3 15 23
		l . .	N. 55° 18′·7	H in space	+ 48° 51′
		Reduction .	10 ·6	h	+ 44 57
		Latitude .	N. 55 29 ·–	Longitude .	W. 3 54

A more accurate Calculation.

	° ′ ″		′ ″		
D . .	+ 19 33 27 } (D)	a . .	+ 23 6		+ 3·14176
a corr .	1			cos D . . .	+ 9·97419
δ . .	+ 18 58 28			(y)	+ 3·11595
(x) . .	+ 0 35 0			(x)	+ 3·32222
		S .	+ 31° 52′·7	tan S . . .	+ 9·79373
$P-\pi$ . .	3·51255			cos S . . .	+ 9·92899
ϖ . . .	9·99903			△	+ 3·39323
P' . .	3·51158				3·51158
		Z .	+ 49° 35′·6	sin Z . . .	+ 9·88165
				tan Z . . .	+ 0·06994

		tan Z . .	+ 0·06994	sin Z . . .	+ 9.88165
	° ′	cos S . .	+ 9.92899	cos S . . .	+ 9·92899
θ	+ 44 55·8	tan θ . .	+ 9·99893		+ 9·81064
δ .	+ 18 58·5	sin θ . .	+ 9·84895	comp. cos h .	+ 0·15020
$\theta + \delta$	+ 63 54·3 .	cos . . .	+ 9·64331	comp. cos l .	+ 0·24480
			+ 0·20564 .	check . . .	+ 0·20564
		tan S . .	+ 9·79373		
	° ′	tan h . .	+ 9·99937		h. m. s.
h	+ 44 57·5 .	cos h . .	+ 9·84980	Greenwich time	3 11 27
		tan ($\theta + \delta$)	+ 0·31000	Equation . .	3 56
		tan l . .	+ 0·15980	H in { time .	+ 3 15 23
		l . . .	N. 55° 18′·7	H in { space .	+ 48° 50′·8
		Reduction .	10 ·6	h	+ 44 57 ·5
		Latitude .	N. 55 29 ·3	Longitude . .	W. 3 53 ·3

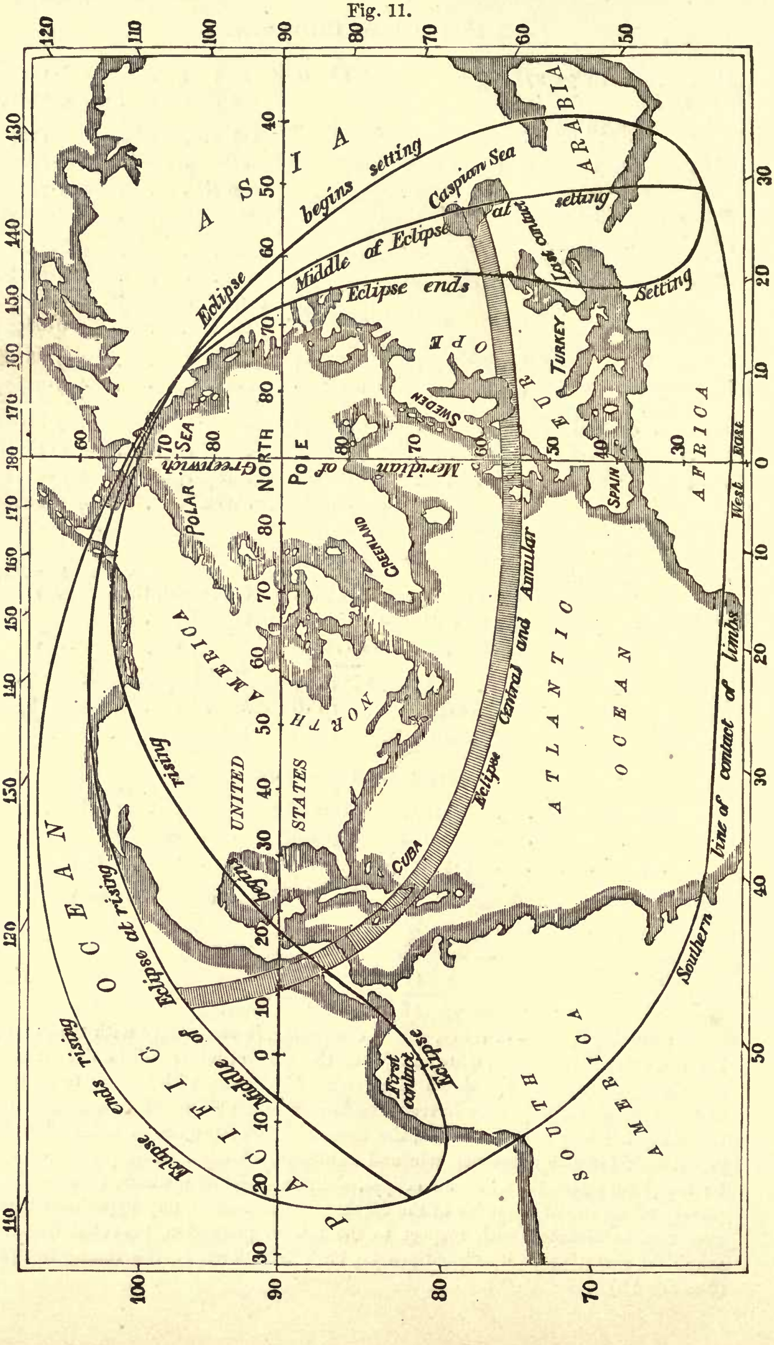
Central Eclipse at Noon.

			h. m. s.
Diff. dec.	3·21245	Time of ☌ . .	2 21 23
P	3·51184	Equation . .	+ 3 56
sin Z	9·70061	Long. in { time .	2 25 19 } W.
Z . . .	+ 30° 8′	Long. in { space	36° 20′ } W.
δ . . .	+ 18 58		
l	N. 49 6		
Reduction . .	11		
Latitude . .	N. 49 17		

By assuming a series of times, and so computing, in conformity with the preceding examples, a series of points on each of the several limits will be determined; and these points being laid down in a geographical map, with respect to latitude and longitude, it will be easy to trace the lines through them. In this manner has the following map been executed, the assumed law of projection being that the parallels of latitude are concentric and equidistant circles. This projection will be found very suitable when an eclipse, as in the present instance, extends completely round one of the poles of the earth. In other cases, any hypothesis whatever may be assumed, with respect to the law of projection, provided the geographical sketching and eclipse-lines be both laid down on the same principle. (See Fig. 11.)

PRINCIPAL LINES FOR THE SOLAR ECLIPSE OF MAY 14–15, 1836

Fig. 11.



PHENOMENA FOR A PARTICULAR PLACE.

I.—Eclipses of the Sun.

The chief objects of determination for any particular place are—

1. For a partial eclipse, its magnitude, and the times of beginning, greatest phase, and ending.

2. For a total eclipse, the times of external and internal contact of limbs, or the times of partial and total beginning and ending.

3. For an annular eclipse, the times of exterior and interior contact of limbs, or the times of partial and annular beginning and ending.

Also, to secure certainty in the observation, it is necessary to determine, in each case, the particular points on the limb of the sun, as related either to the vertical or a circle of declination, where these contacts take place; and hence the general configuration of the eclipse.

We first proceed to find expressions for calculating, at any time, the apparent relative position of the two bodies, and the augmentation of the semi-diameter of the moon. The parallax in altitude depends on the Eq. (8) or (9), page 336. It will here be necessary to investigate the effects which this parallax will produce in the right ascension and declination of the moon. These might be accurately determined by the theory of the small variations of spherical triangles, but not quite so simply as in the following manner:—Assume, as before,

l, the geocentric latitude of the place;
$R. A.$, the true right ascension of the moon;
D, the true declination of the moon, $+$ north, $-$ south;
h, the true hour angle of the moon, $+$ west, $-$ east;
r, the distance of the centres of the earth and moon.

Then if, from the earth's centre, we take

x, on the intersection of the planes of the meridian and equator, $+$ towards upper meridian;
y, in the plane of the equator, $+$ west, $-$ east;
z, parallel to the earth's axis, $+$ north, $-$ south;

we shall have, for the position of the moon,

$$x = r \cos D \cos h, \qquad y = r \cos D \sin h, \qquad z = r \sin D;$$

and, for the position of the observer,

$$(x) = \rho \cos l, \qquad (y) = 0, \qquad (z) = \rho \sin l.$$

Thus the position of the moon, in relation to the observer as an origin, will be

$$x' = x - (x) = r \cos D \cos h - \rho \cos l;$$
$$y' = y - (y) = r \cos D \sin h;$$
$$z' = z - (z) = r \sin D - \rho \sin l;$$

and hence, D', h' denoting the apparent declination and hour angle, and r' the distance of the moon from the observer, we shall have

$$x' = r' \cos D' \cos h' = r \cos D \cos h - \rho \cos l;$$
$$y' = r' \cos D' \sin h' = r \cos D \sin h;$$
$$z' = r' \sin D' \qquad = r \sin D - \rho \sin l.$$

Therefore, as $\cot h' = \frac{x'}{y'}$, $\tan D' = \frac{z'}{y'} \sin h'$, $\frac{1}{r} = \sin P$, we find

$$\left.\begin{aligned} \cot h' &= \cot h - \frac{\rho \sin P \cos l}{\cos D \sin h} \\ \tan D' &= \left(1 - \frac{\rho \sin P \sin l}{\sin D}\right) \frac{\sin h'}{\sin h} \tan D \\ \text{or} \qquad \cot h - \cot h' &= \left(\frac{\rho \sin P}{\cos D \sin h}\right) \cos l \\ \frac{\tan D}{\sin h} - \frac{\tan D'}{\sin h'} &= \left(\frac{\rho \sin P}{\cos D \sin h}\right) \sin l \end{aligned}\right\} \quad . \; . \; (1)$$

which present a direct method of calculating the apparent position of the moon, at any time, from that of the true. The former of these equations is evidently subservient to the other, and must necessarily be computed first. As the calculation of these expressions will, in general, require seven places of figures, it will be more convenient to determine the simple effects of the parallax, or the small differences $A.R. - A.R.'$, $D - D'$, for which other expressions may be derived from them. Let $A.R. - A.R.' = h' - h = \triangle h$, and $D - D' = \triangle D$; then by multiplying the equation

$$\cot h - \cot h' = \frac{\rho \sin P \cos l}{\cos D \sin h}$$

by $\sin h \sin h'$, the left-hand member will become $\sin (h' - h)$ or $\sin \triangle h$.

$$\therefore \sin \triangle h = \frac{\rho \sin P \cos l}{\cos D} \sin h'.$$

Again we have

$$\frac{\tan D}{\sin h} - \frac{\tan D'}{\sin h'} = \frac{\rho \sin P \sin l}{\cos D \sin h}.$$

But

$$\frac{\tan D}{\sin h} - \frac{\tan D'}{\sin h'} = \frac{\tan D - \tan D'}{\sin h} + \left(\frac{1}{\sin h} - \frac{1}{\sin h'}\right) \tan D';$$

$$= \frac{\sin (D - D')}{\sin h \cos D \cos D'} + \frac{\sin h' - \sin h}{\sin h \sin h'} \tan D';$$

$$= \frac{\sin \triangle D}{\sin h \cos D \cos D'} + \frac{2 \sin \frac{1}{2} \triangle h \cos (h + \frac{1}{2} \triangle h)}{\sin h \sin h'} \tan D'.$$

Equate this with $\frac{\rho \sin P \sin l}{\cos D \sin h}$, and we find

$$\frac{\sin \triangle D}{\cos D \cos D'} = \frac{\rho \sin P \sin l}{\cos D} - \frac{2 \sin \frac{1}{2} \triangle h \cos (h + \frac{1}{2} \triangle h)}{\sin h'} \cdot \frac{\sin D'}{\cos D'}$$

But $2 \sin \frac{1}{2} \triangle h = \frac{\sin \triangle h}{\cos \frac{1}{2} \triangle h} = \frac{\rho \sin P \cos l}{\cos D} \cdot \frac{\sin h'}{\cos \frac{1}{2} \triangle h}$.

Substitute this value and multiply by $\cos D \cos D'$ and we deduce

$$\sin \triangle D = \rho \sin P \left[\sin l \cos D' - \cos l \sin D' \frac{\cos (h + \frac{1}{2} \triangle h)}{\cos \frac{1}{2} \triangle h} \right].$$

We shall therefore have, for the parallax of the hour angle, and that of the declination,

$$\left.\begin{aligned} \sin \triangle h &= \frac{(\rho \cos l) \sin P}{\cos D} \sin h' \\ \sin \triangle D &= \sin P \left[(\rho \sin l) \cos D' - (\rho \cos l) \sin D' \frac{\cos (h + \frac{1}{2} \triangle h)}{\cos \frac{1}{2} \triangle h} \right] \end{aligned}\right\} \quad . \; . \; (2)$$

These are still however not adapted for direct calculation, since they involve the apparent quantities h', D', which it is our object to determine. The only use that can be made of them is, first to use the true quantities, in order to get the parallaxes and apparent values approximately, and then to repeat the operation. To avoid this difficulty, substitute in the former $h + \triangle h$ instead of h', and in the latter put $D - \triangle D$ instead of D', and we get, by expansion,

$$\sin \triangle h = \frac{\rho \cos l \sin P}{\cos D} (\sin h \cos \triangle h + \cos h \sin \triangle h);$$

$$\sin \triangle D = \rho \sin P \cos \triangle D \left[\sin l \cos D - \cos l \sin D \frac{\cos (h + \frac{1}{2} \triangle h)}{\cos \frac{1}{2} \triangle h} \right]$$

$$+ \rho \sin P \sin \triangle D \left[\sin l \sin D + \cos l \cos D \frac{\cos (h + \frac{1}{2} \triangle h)}{\cos \frac{1}{2} \triangle h} \right].$$

Divide these by $\cos \triangle h$, $\cos \triangle D$, respectively, and solve for $\tan \triangle h$ and $\tan \triangle D$, and we find

$$\tan \triangle h = \frac{\left(\frac{\rho \cos l \sin P}{\cos D}\right) \sin h}{1 - \left(\frac{\rho \cos l \sin P}{\cos D}\right) \cos h} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (3)$$

$$\left.\begin{aligned} \tan \triangle D &= \frac{\rho \sin P \left[\sin l \cos D - \cos l \sin D \frac{\cos (h + \frac{1}{2} \triangle h)}{\cos \frac{1}{2} \triangle h} \right]}{1 - \rho \sin P \left[\sin l \sin D + \cos l \cos D \frac{\cos (h + \frac{1}{2} \triangle h)}{\cos \frac{1}{2} \triangle h} \right]} \\ &= \frac{(\rho \sin l \sin P) \cos D \left[1 - \frac{\tan D}{\tan l} \cdot \frac{\cos (h + \frac{1}{2} \triangle h)}{\cos \frac{1}{2} \triangle h} \right]}{1 - (\rho \sin l \sin P) \sin D \left[1 + \frac{1}{\tan l \tan D} \cdot \frac{\cos (h + \frac{1}{2} \triangle h)}{\cos \frac{1}{2} \triangle h} \right]} \end{aligned}\right\} \quad . \; (4)$$

These expressions are all of them perfectly rigorous, and better suited to calculation than they would appear at first sight. The process of the calculation, in which five places of figures will be sufficient, is more detailed in the following equations:

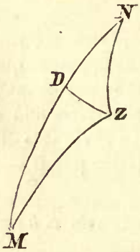
$$\mathrm{n} = \frac{(\rho \cos l) \sin P}{\cos D}; \qquad \tan \triangle h = \frac{\mathrm{n} \sin h}{1 - \mathrm{n} \cos h} \quad . \; . \quad (5)$$

$$\left.\begin{array}{ll} c = (\rho \sin l) \sin P; & k = \dfrac{\cos (h + \frac{1}{2} \triangle h)}{\cos \frac{1}{2} \triangle h} . \cot l \\ n_1 = k \tan D; & n_2 = \dfrac{k}{\tan D} \\ \multicolumn{2}{c}{\tan \triangle D = \dfrac{c \cos D (1 - n_1)}{1 - c \sin D (1 + n_2)}} \end{array}\right\} \quad . \; . \qquad (6)$$

The expression (4) for tan $\triangle D$ may, however, be neatly resolved by means of a spherical triangle as follows:

Assume

Fig. 9.



$$\cos (h) = \frac{\cos (h + \frac{1}{2} \triangle h)}{\cos \frac{1}{2} \triangle h} \quad . \; . \; (a)$$

(h) being very nearly equal to $h + \frac{1}{2} \triangle h$. And let N be the north pole, Z the central zenith, and M the moon; then $NM = 90° - D$, $NZ = 90° - l$, and the $\angle N = h$. Without changing these values of NM, NZ, let us suppose the hour angle N to become increased to the value of (h); and with the triangle so constituted suppose the altitude of the moon to be ε, so that $ZM = 90° - \varepsilon$; then the spherical relations

$$\sin ZM \cos M = \cos NZ \sin NM - \sin NZ \cos NM \cos N,$$
$$\cos ZM = \cos NZ \cos NM + \sin NZ \sin NM \cos N,$$

will give

$$\cos \varepsilon \cos M = \sin l \cos D - \cos l \sin D \cos (h)$$
$$= \sin l \cos D - \cos l \sin D \frac{\cos (h + \frac{1}{2} \triangle h)}{\cos \frac{1}{2} \triangle h},$$
$$\sin \varepsilon = \sin l \sin D + \cos l \cos D \cos (h)$$
$$= \sin l \sin D + \cos l \cos D \frac{\cos (h + \frac{1}{2} \triangle h)}{\cos \frac{1}{2} \triangle h}.$$

Comparing these with the former expression of (4), we have therefore

$$\tan \triangle D = \frac{(\rho \sin P) \cos \varepsilon}{1 - (\rho \sin P) \sin \varepsilon} . \cos M \quad . \; . \; . \qquad (f)$$

Before this can be used the angles M and ε must be determined.

Draw ZD perpendicular to MN, and by spherics,

$$\tan ND = \tan NZ \cos N \quad . \; . \; . \; . \; . \; . \; . \qquad (b)$$

$$\sin MD \tan M = \tan ZD = \sin ND \tan N;$$

$$\therefore \tan M = \frac{\sin ND}{\sin MD} \tan N \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (c)$$

$$\tan MZ = \frac{\tan MD}{\cos M}, \text{ or } \cot MZ = \cot MD \cos M \quad . \; . \; . \; (d)$$

Also by (c)

$$\frac{\sin ND}{\sin MD} = \frac{\tan M}{\tan N} = \frac{\cos N}{\cos M} . \frac{\sin M}{\sin N} = \frac{\cos N \sin NZ}{\cos M \sin MZ} \quad . \; . \qquad (e)$$

Let now $ND = \theta$, and $MD = MN - \theta = 90° - (\theta + D)$; and the equations (*a*), (*b*), (*c*), (*d*), (*e*), (*f*), will give the following:

$$\left.\begin{aligned}
\cos(h) &= \frac{\cos(h + \frac{1}{2}\Delta h)}{\cos\frac{1}{2}\Delta h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a) \\
\tan\theta &= \cot l \cos(h) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b) \\
\tan M &= \frac{\sin\theta}{\cos(\theta + D)}\tan(h) \quad . \quad . \quad . \quad . \quad . \quad . \quad (c) \\
\tan\varepsilon &= \tan(\theta + D)\cos M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d) \\
\frac{\sin\theta}{\cos(\theta + D)} &= \frac{\cos(h)\cos l}{\cos M\cos\varepsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e) \\
\tan\Delta D &= \frac{(\rho\sin P)\cos\varepsilon}{1 - (\rho\sin P)\sin\varepsilon}\cos M \quad . \quad . \quad . \quad . \quad (f)
\end{aligned}\right\} \quad . \quad (7)$$

in which the equation (*e*) is used as a check on the preceding computations. This check affords a good security to the accuracy of the work, and gives to these equations a decided preference over those of (6), although a trifle more perhaps in point of calculation. They have also another advantage, inasmuch as M may be considered as the parallactic angle, and ε the altitude of the moon; the former of these is useful in determining the position of the line joining the centres of the two bodies in relation to the vertical, and the other is useful in finding the augmentation of the moon's semi-diameter, which we shall now consider.

If s' denote the moon's apparent semi-diameter, and s her true semi-diameter as seen from the centre of the earth, the actual semi-diameter of the moon will be represented by both $r \sin s$, and $r' \sin s'$; also, if a perpendicular be drawn from the centre of the moon upon the radius ρ produced, this perpendicular will be represented by both $r \sin Z$, and $r' \sin Z'$. We must therefore have $\frac{\sin s'}{\sin s} = \frac{\sin Z'}{\sin Z}$.

Let M be the true position of the moon, in the preceding figure, and $\sin ZM \sin \angle NZM = \sin NM \sin N$ will be $\sin Z \sin \angle NZM = \cos D \sin h$; for the apparent position of the moon the angle NZM will remain the same, and $\sin Z' \sin \angle NZM = \cos D' \sin h'$.

$$\therefore \frac{\sin Z'}{\sin Z} = \frac{\cos D'}{\cos D} \cdot \frac{\sin h'}{\sin h}.$$

Also, by means of the equations (8) and (9), page 336,

$$\frac{\sin Z'}{\sin Z} = \frac{\rho \sin P \sin Z'}{\rho \sin P \sin Z} = \frac{\sin z}{\rho \sin P \sin Z} = \frac{\cos z}{\rho \sin P \sin Z}\tan z = \frac{\cos z}{1 - \rho \sin P \cos Z}$$

$$\therefore \frac{\sin s'}{\sin s} = \frac{\sin Z'}{\sin Z} = \frac{\cos D'}{\cos D} \cdot \frac{\sin h'}{\sin h} = \frac{\cos z}{1 - \rho \sin P \cos Z} \quad . \quad . \quad . \quad . \quad (8)$$

All the preceding formulæ are strict in theory. It now remains to consider what allowances may be made and what facilities given in their actual calculation. In the first place the value of $\cos \frac{1}{2} \Delta h$ may be safely assumed equal to *unity*, and may therefore be rejected in the equations (2), (4), (6), and (7), so that $(h) = h + \frac{1}{2} \Delta h$; it may be shown that this supposition cannot involve an error of more than $0''.03$ in the value of ΔD.

Also, as the arcs P, $\triangle h$, $\triangle D$, are small, we must have very nearly

$$\frac{\sin P}{P} = \sin 1'' = [4.68557], \quad \frac{\tan \triangle h}{\triangle h} = \frac{\tan \triangle D}{\triangle D} = \tan 1'' = [4.68557],$$

where P, $\triangle h$, $\triangle D$, denote respectively the numbers of seconds they contain. These equations may be made more exact, for the limits between which the angles are always comprised, by adopting numbers differing a little from $\sin 1''$ and $\tan 1''$; thus, by assuming

$$\frac{\sin P}{P} = [4.68555], \qquad \frac{\tan \triangle h}{\triangle h} = [4.68561].$$

The first supposition will not in any case involve an error exceeding that of $0''.05$ in the value of P, nor the second an error of more than $0''.1$ in the value of $\triangle h$, and these are much too small to merit attention; the latter assumption applies equally the same to $\triangle D$.

Thus we shall have $(h) = h + \frac{1}{2} \triangle h$, $\sin P = [4.68555]\ P$, $\triangle h = [5.31439] \tan \triangle h$, $\triangle D = [5.31439] \tan \triangle D$; also, $\triangle h = \triangle a$, the parallax in right ascension. The equations (3) and (7) may therefore be commodiously arranged as follows:

$$\left.\begin{array}{lll} & c = [4.68555]\ \rho; & \\ A = cP; & m = A \cos l; & k = \dfrac{m}{\cos D} \\ n = k \cos h; & \triangle a = [5.31439]\ \dfrac{k \sin h}{1 - n}; & \end{array}\right\} \quad . \quad . \quad . \ (9)$$

By taking h less than 180°, positively or negatively, $\triangle a$ will have the same sign as h.

$$\left.\begin{array}{ll} & (h) = h + \frac{1}{2} \triangle a; \\ \tan \theta = \cos (h) \cot l; & G = \cos (h) \cos l \\ \tan M = \dfrac{\sin \theta}{\cos (\theta + D)} \tan (h); & \tan \varepsilon = \tan (\theta + D) \cos M \\ B = \cos M \cos \varepsilon, & \text{check} \ . \ . \ . \ \dfrac{\sin \theta}{\cos (\theta + D)} = \dfrac{G}{B} \\ n_1 = A \sin \varepsilon; & \triangle D = [5.31439]\ \dfrac{AB}{1 - n_1} \end{array}\right\} \quad . \ . \ (10)$$

The auxiliary arc θ may be taken out in the first quadrant, $+$ or $-$; calling 0° to 180° the first semicircle, and 180° to 360° or 0° to $-$ 180° the second semicircle, the parallactic angle M must be taken out in the same semicircle with h; and $\triangle D$ will have the same sign as $\cos M$.

It will appear by the preceding investigations that the values of $\triangle a$, $\triangle D$, so deduced, are the quantities to be *subtracted* from the true values of *A.R.*, D, to get the apparent.

As the number n is always very small, the values of comp. log. $(1 - n)$ to the fifth place of figures may be comprised in the following useful Table under the title of *Correction of Log. Parallax,* and conveniently taken out with the nearest third figure of the argument.

Correction of Log. Parallax.

Argument: log. n.

Log n	Corr.	Log n	Corr.	Log n	Corr.	Log n	Corr.	Log n	Corr.
5.00	0	7.100	54	7.400	109	7.700	218	8.000	436
.10	0	.110	55	.410	112	.710	223	.010	447
.20	1	.120	57	.420	114	.720	229	.020	457
.30	1	.130	58	.430	117	.730	234	.030	468
.40	1	.140	60	.440	120	.740	240	.040	479
.50	1	.150	61	.450	123	.750	245	.050	490
.60	2	.160	63	.460	125	.760	251	.060	501
.70	2	.170	64	.470	128	.770	257	.070	513
.80	2	.180	66	.480	131	.780	263	.080	525
.90	3	.190	68	.490	134	.790	269	.090	537
6.00	4	.200	69	.500	137	.800	275	.100	550
.10	6	.210	71	.510	141	.810	281	.110	563
.20	7	.220	72	.520	144	.820	288	.120	576
.30	9	.230	74	.530	148	.830	294	.130	590
.40	11	.240	76	.540	151	.840	302	.140	604
.50	14	.250	77	.550	155	.850	308	.150	618
.60	17	.260	79	.560	158	.860	315	.160	632
.70	22	.270	81	.570	162	.870	323	.170	647
.80	27	.280	83	.580	165	.880	331	.180	663
.90	34	.290	85	.590	169	.890	338	.190	678
7.00	43	.300	87	.600	173	.900	346	.200	694
7.000	43	.310	89	.610	177	.910	355	.210	710
.010	44	.320	91	.620	181	.920	363	.220	727
.020	46	.330	93	.630	186	.930	371	.230	744
.030	47	.340	95	.640	191	.940	379	.240	761
.040	48	.350	98	.650	195	.950	388	.250	779
.050	49	.360	100	.660	199	.960	398	8.260	798
.060	50	.370	102	.670	204	.970	407		
.070	51	.380	104	.680	209	.980	417		
.080	52	.390	107	.690	213	7.990	427		
.090	53	7.400	109	7.700	218	8.000	436		
7.100	54								

This correction is additive when n is positive, and subtractive when n is negative. For the parallax in declination it will always be additive if the moon be above the horizon.

For the augmentation of the moon's semi-diameter we may assume $\cos z = 1$ and $Z = 90° - \varepsilon$, so that

$$\frac{s'}{s} = \frac{1}{1 - \rho \sin P \sin \varepsilon} = \frac{1}{1 - n_1};$$

n_1 being the number which enters into the computation of ΔD. Hence

$$s' = \frac{s}{1 - n_1} = \frac{[9.43537]\,P}{1 - n_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This and the last formulæ for Δa, ΔD, entirely preclude the necessity of having recourse to a table of the sines and tangents of small arcs, and possess much uniformity and simplicity in their application.

To get the relative parallax of the moon with respect to the sun, we must use $P - \pi$, instead of P. If, therefore, P' denote the value of $\rho\,(P - \pi)$, or the relative horizontal parallax reduced to the latitude of the place, we must use $\sin P'$, instead of $\rho \sin P$, in the preceding formulæ.

The determination of the apparent relative positions of the centres of the two bodies, as well as the augmentation of the semi-diameter of the moon, at any time, has now been reduced to a practical and expeditious set of formulæ. A series of these apparent positions of the moon, with respect to that of the sun, will trace out her apparent relative orbit; and the contact of limbs will evidently take place when the apparent distance of the centres becomes equal to the sum or difference of the semi-diameter of the sun and the augmented semi-diameter of the moon. For a distance equal to the sum of these semi-diameters we shall have partial beginning or ending; for a distance equal to their difference we shall have

$$\left.\begin{matrix}\text{total}\\ \text{annular}\end{matrix}\right\}\text{ beginning or ending, when } s' \left\{\begin{matrix} > \\ < \end{matrix}\right\} \sigma.$$

Since the hour angle of the bodies is subject to the rapid variation of nearly 15° per hour, the effect produced by parallax will be of so irregular a nature as to give a decided curvature to the apparent relative orbit of the moon. This curvature will be more strongly characterized when the eclipse takes place at some distance from the meridian or near to the horizon; and the apparent relative hourly motion of the moon, even during the short interval of the duration of the eclipse, will, through the same irregular influence, experience considerable variation. These circumstances will, in some measure, vitiate any results deduced in the usual manner, by supposing the portion of the orbit described during the eclipse to be a straight line, and using the relative motion at the time of apparent conjunction as a uniform quantity. The method we are about to pursue is very simple, and consists in assuming any time within the eclipse, and computing for this time the relative positions and motion of the bodies, and thence finding, without any reference whatever, either to the time of the middle of the eclipse or to the time of conjunction, the times of beginning, greatest phase, and ending, and the relative positions of the bodies at these times. The nearer the assumed time is to the time of the greatest phase, the more accurately will the time of that phase be determined; and, similarly, the nearer that time is to the time of beginning or ending, the more certainty will attach to the determination.

To find the apparent relative motion of the moon, we must first determine the variation which takes place in the parallax. For this, take the equations (2), p. 379, viz.:

$$\sin \Delta a = \sin \Delta h = \frac{\sin P' \cos l}{\cos D} \sin h',$$

$$\sin \Delta D = \sin P' \left[\sin l \cos D' - \cos l \sin D' \frac{\cos (h + \frac{1}{2} \Delta h)}{\cos \frac{1}{2} \Delta h}\right];$$

or, substituting small arcs instead of their sines,

$$\Delta a = P' \frac{\cos l}{\cos D} \sin h',$$

$$\Delta D = P' \left[\sin l \cos D' - \cos l \sin D' \frac{\cos (h + \frac{1}{2} \Delta h)}{\cos \frac{1}{2} \Delta h}\right].$$

Since a portion of the apparent disk of the moon is projected on that of the sun, the apparent declination D' can differ very little from δ. As the hourly variations of these small quantities are only required approximately, we may therefore use δ instead of D' and neglect $\triangle h$, so as to have

$$\triangle a = P' \frac{\cos l}{\cos D} \sin h,$$

$$\triangle D = P' (\sin l \cos \delta - \cos l \sin \delta \cos h);$$

which expressions, though rough values of $\triangle a$, $\triangle D$, will give their hourly variations pretty accurately. For these, observing that h is the only quantity which, by its rapid variation, has any sensible influence on these values, we have by differentiation,

$$\frac{d(\triangle a)}{dt} = \left(P' \frac{dh}{dt} \sin 1''\right) \frac{\cos l}{\cos D} \cos h,$$

$$\frac{d(\triangle D)}{dt} = \left(P' \frac{dh}{dt} \sin 1''\right) \cos l \sin \delta \sin h.$$

But by the equations (9),

$$m = [4.68555]\, P' \cos l,$$

$$n = [4.68555]\, P' \frac{\cos l}{\cos D} \cos h.$$

Substitute, therefore,

$$P' \frac{\cos l}{\cos D} \cos h = [5.31445]\, n,$$

$$P' \cos l = [5.31445]\, m;$$

and we get

$$\frac{d(\triangle a)}{dt} = [5.31445] \left(\frac{dh}{dt} \sin 1''\right) n,$$

$$\frac{d(\triangle D)}{dt} = [5.31445] \left(\frac{dh}{dt} \sin 1''\right) m \sin \delta \sin h.$$

If we adopt 14° 29′ as a mean value of $\frac{dh}{dt}$, we shall have $\frac{dh}{dt} \sin 1'' = [9.40274]$ and $[5.31445] \left(\frac{dh}{dt} \sin 1''\right) = [4.71719]$ or $[4.7172]$. Therefore, if (δ), the value of the sun's declination at the time of the middle of the eclipse, be adopted in the value of $\frac{d(\triangle D)}{dt}$, we may form the constants,

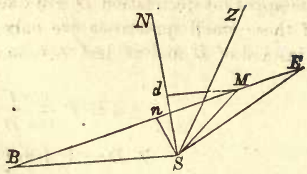
$$\left.\begin{array}{l} Q_1 = [4.7172], \\ Q_2 = [4.7172]\, m \sin (\delta) \end{array}\right\} \quad \ldots\ldots\ldots \quad (12)$$

and then, using $\triangle a_1$, $\triangle D_1$ in place of $\frac{d(\triangle a)}{dt}$, $\frac{d(\triangle D)}{dt}$, we shall have

$$\left.\begin{array}{l} \triangle a_1 = Q_1 n, \\ \triangle D_1 = Q_2 \sin h \end{array}\right\} \quad \ldots\ldots \quad (13)$$

which offer a simple calculation.

Fig. 10.



Let now, at any assumed time within the duration of the eclipse, S and M be the apparent positions of the centres of the sun and moon; and $B\,M\,E$ an arc of a great circle coinciding with the relative direction of the moon's motion at that time, which arc we shall first adopt in place of the curvilinear orbit actually described. On the circle of declination $S\,N$, demit the great circle perpendicular $M\,d$, and suppose B and E to be the positions of the moon at the respective times of partial beginning and ending of the eclipse, and n the middle point. Assume $S\,B = S\,E = s' + \sigma = \triangle'$, $S\,d = x$, $d\,M = y$, $S\,M = W$ $S\,n = n$, $\angle\,N\,S\,M = S$, $\angle\,B\,M\,d = \angle\,d\,S\,n = \iota$, and the $\angle\,B\,S\,n = \angle\,E\,S\,n = \omega$. Also, for simplicity, let x_1, y_1 denote the hourly variations of x and y.

In determining the value of x we shall require the a correction, which will reduce the declination of the point M to that of d. This correction is shown in a table at p. 342; but, as this small correction may be wanted more accurately than can be obtained from that table, we shall here give some factors for its determination, from which, in fact, the table alluded to has been derived The correction will resolve as follows:

$$\tan\,(D) = \frac{\tan D}{\cos a};$$

$$\therefore \tan\,[(D) - D] = \frac{\dfrac{\tan D}{\cos a} - \tan D}{1 + \dfrac{\tan^2 D}{\cos a}} = \frac{\tan D\,(1 - \cos a)}{\cos a + \tan^2 D}.$$

Or, supposing $\cos a = 1$ in the denominator,

$$\tan\,[(D) - D] = \frac{\tan D\,(1 - \cos a)}{1 + \tan^2 D} = \sin 2\,D \sin^2 \frac{a}{2}.$$

Suppose, now, a to be expressed numerically in *minutes*, and $(D) - D$ in *seconds;* then

$$\tan\,[(D) - D] = [(D) - D] \sin 1'';$$

$$\sin \frac{a}{2} = \frac{a}{2} \sin 1' = (30 \sin 1'')\,a.$$

Therefore, by substitution, we find

$$(D) - D = (900 \sin 1'' \sin 2\,D)\,a^2.$$

Consequently, assuming

$$F = 90000 \sin 1'' \sin 2\,D = [9.63982] \sin 2\,D,$$

we shall have

$$a \text{ corr.} = (D) - D = F\,.\left(\frac{a}{10}\right)^2.$$

The value of F, argument D, is contained in the following small table

Factor F for a correction.

D	F	D	F	D	F
°		°		°	
0	·000	10	·149	20	·280
1	·015	11	·164	21	·292
2	·030	12	·178	22	·303
3	·046	13	·191	23	·314
4	·061	14	·205	24	·324
5	·076	15	·218	25	·334
6	·091	16	·231	26	·344
7	·106	17	·244	27	·353
8	·120	18	·256	28	·362
9	·135	19	·268	29	·370
10	·149	20	·280		

a corr. in *seconds* $= F \cdot \left(\frac{a}{10}\right)^2$

a denoting the number of *minutes* it contains.

From what has preceded, it is evident that $a' = a - \triangle a$, is the apparent difference of the right ascensions of the bodies, and that $D' = D - \triangle D$ is the apparent declination of the moon; and that

$$\left.\begin{array}{l} x = [D' + (a - \triangle a) \text{ corr.}] - \delta \\ y = [a \quad - \triangle a] \cos D' \end{array}\right\} \quad \ldots\ldots\ldots \quad (14)$$

and consequently also

$$\left.\begin{array}{l} x_1 = D_1 - \triangle D_1 \\ y_1 = (a_1 - \triangle a_1) \cos D' \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (15)$$

Moreover, the figure occupying so small a portion of the sphere, and being composed of arcs of great circles, we may, without any appreciable error, treat these arcs as straight lines; thence we shall obviously have

$$\left.\begin{array}{l} \tan S = \frac{y}{x}, \qquad W = \frac{y}{\sin S} = \frac{x}{\cos S} \\ \cot \iota = \frac{y_1}{x_1}, \\ \text{Hourly motion in the orbit} = \frac{y_1}{\cos \iota} \\ n = W \cos (S + \iota), \qquad \cos \omega = \frac{n}{\triangle'} \end{array}\right\} \quad \ldots\ldots \quad (16)$$

Again, in the triangles BSM, ESM,

$$\angle BSM = \omega + (S + \iota), \qquad \angle ESM = \omega - (S + \iota);$$

and consequently, by plane trigonometry,

$$BM = \frac{W}{\cos \omega} \sin [\omega + (S + \iota)], \qquad EM = \frac{W}{\cos \omega} \sin [\omega - (S + \iota)],$$

$$nM = W \sin (S + \iota).$$

With the above hourly motion in the orbit we shall therefore have

$$\text{Time of describing}\begin{cases} BM = \dfrac{W\cos\iota}{y_1\cos\omega}\sin\left[\omega + (S+\iota)\right], \\ nM = \dfrac{W\cos\iota}{y_1}\sin(S+\iota), \\ EM = \dfrac{W\cos\iota}{y_1\cos\omega}\sin\left[\omega - (S+\iota)\right]. \end{cases}$$

Let, now, t_1, t_2, be corrections to be applied to the time assumed to get the times of beginning and ending, and (t) the correction for the time of the greatest phase. Then we have evidently

$$\left\{\begin{matrix} t_1 \\ (t) \\ t_2 \end{matrix}\right\} = \text{the time of describing} \left\{\begin{matrix} BM \\ nM \\ EM \end{matrix}\right\} \text{with a} \left\{\begin{matrix} \text{negative} \\ \text{negative} \\ \text{positive} \end{matrix}\right\} \text{sign.}$$

To have these times expressed in seconds, assume

$$c = \frac{W\cos\iota}{y_1\cos\omega} \times 3600'' = \frac{W\cos\iota}{y_1} \cdot \frac{[3.55630]}{\cos\omega} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and then we shall derive

$$t_1 = c\sin\left[-(S+\iota) - \omega\right], \qquad t_2 = c\sin\left[-(S+\iota) + \omega\right],$$

$$(t) = c\cos\omega\sin\left[-(S+\iota)\right],$$

and hence

$$\text{The time of} \left\{\begin{matrix} \text{beginning} \\ \text{greatest phase} \\ \text{ending} \end{matrix}\right\} = \text{assumed time} + \left\{\begin{matrix} c\sin\left[-(S+\iota) - \omega\right] \\ c\cos\omega\sin\left[-(S+\iota)\right] \\ c\sin\left[-(S+\iota) + \omega\right] \end{matrix}\right\} \quad (18)$$

It has been observed, that any one of these values will be the more to be depended on the more nearly it approximates to the assumed time. Thus, if the assumed time be within ten minutes or so of the end of the eclipse, the point M will approximate so closely to the point E, that no sensible error can arise by supposing the small portion ME of the orbit to be a straight line, and to be passed over by the moon with a uniform motion. This circumstance renders it advisable, in the first instance, to take the assumed time near to the time of the middle of the eclipse, so as to give a good result for the time of the greatest phase, and results for the times of beginning and ending, which may be nearly equally relied on. Such a computation will be sufficiently exact for the usual purposes of prediction. When the time of beginning or ending is wanted to great minuteness to compare with observation, it will only be necessary to repeat the operation for a time assumed as near as convenient to the first determination, which will mostly give within a fractional part of a second of the true theoretical result; a degree of accuracy, however, seldom wished for, and quite unsupported by the present state of the lunar theory.

To fix on a time near to the middle of the eclipse for the radical computation, one of the most simple expedients will be to determine roughly the time of the apparent conjunction.

We shall now briefly consider the apparent positions of the moon, as related to the sun's centre.

It is clear that S is the angle of position of the moon's centre from the north towards the east, at the time assumed; also that the angle $NSB = \omega + \iota$ is the similar angle of position from the north towards the west at the time of beginning; and that the angle $NSE = \omega - \iota$ is the angle of position from the north towards the east at the time of ending; and that the angle $NSn = \iota$ is the same angle towards the west at the time of the greatest phase. Therefore, by estimating all these angles towards the east we shall have

$$\text{At}\left\{\begin{array}{c}\text{beginning}\\ \text{greatest phase}\\ \text{ending}\end{array}\right\} \angle \text{ of ☽'s centre from N. towards E.} = \left\{\begin{array}{l}(-\iota)-\omega\\ (-\iota)\\ (-\iota)+\omega\end{array}\right\} \qquad (19)$$

In the computation of the parallax in declination, we find an angle M, which in practice may be supposed to be the angle NSZ for the assumed time, the zenith Z being reckoned towards the east; consequently, at this time we shall have $S - M$ for the angle of position of the moon's centre from the zenith towards the east. At any other time the parallactic angle M for the latitude of Greenwich may be taken from the following table, arguments the corresponding apparent time and the sun's declination. This table, for any other place, may be computed by formulæ, such as at page 381, viz.:

$$\tan\theta = \cot l \cos h, \qquad \tan M = \frac{\sin\theta}{\cos(\theta+\delta)}\tan h,$$

h being the angle answering to the apparent time.

Those who may be engaged in the computation of eclipses, for any particular places, will find considerable facility in the formation of similar tables.

For an occultation of a star by the moon, the argument, instead of the apparent time, will be the star's hour angle, or the sidereal time *minus* the star's right ascension. In this case the required positions will be those of the star with respect to the moon's centre, which will therefore be different from the angles of position for a solar eclipse, in which the moon's centre is referred to that of the sun. The angular positions of the contacts at immersion and emersion will consequently be determined in the same way as for an eclipse of the sun, and will be estimated in the opposite directions. Thus, for an occultation,

$$\text{At}\left\{\begin{array}{c}\text{immersion}\\ \text{emersion}\end{array}\right\} \angle \text{ of } \ast \text{ from N. towards E.} = \left\{\begin{array}{l}(180^\circ-\iota)-\omega\\ (180^\circ-\iota)+\omega\end{array}\right\}$$

And so must 180° be applied to the other angles of position, as expressed for a solar eclipse: this will make the expressions for the direct images of occultations the same as those for the inverted images of eclipses of the sun, in estimating the contacts either from the north point or from the vertex.

Parallactic Angles for the Latitude of Greenwich.

(*same sign as h*)

Arguments: Apparent Hour Angle and *Declination.*

Dec. North.	Hour Angle h.														
	° 0	° 10	° 20	° 30	° 40	° 50	° 60	° 70	° 80	° 90	° 100	° 110	° 120	° 130	° 140
°	°	°	°	°	°	°	°	°	°	°	°	°	°	°	°
0	0	8	15	22	27	31	35	37	38	39	38	37	35	31	27
1	0	8	15	22	27	32	35	37	38	39	38	37	34	31	27
2	0	8	16	22	28	32	35	37	38	39	38	37	34	31	27
3	0	8	16	22	28	32	35	37	38	39	38	36	34	31	26
4	0	8	16	23	28	32	35	37	38	39	38	36	34	31	26
5	0	9	16	23	28	33	36	38	39	39	38	36	34	30	26
6	0	9	17	23	29	33	36	38	39	39	38	36	34	30	26
7	0	9	17	24	29	33	36	38	39	39	38	36	34	30	26
8	0	9	17	24	29	34	36	38	39	39	38	36	33	30	25
9	0	9	17	24	30	34	37	38	39	39	38	36	33	30	25
10	0	9	18	25	30	34	37	39	39	39	38	36	33	30	25
11	0	9	18	25	31	35	37	39	39	39	38	36	33	29	25
12	0	10	18	25	31	35	38	39	40	39	38	36	33	29	25
13	0	10	19	26	31	35	38	39	40	39	38	36	33	29	25
14	0	10	19	26	32	36	38	40	40	39	38	36	33	29	25
15	0	10	19	27	32	36	39	40	40	39	38	36	33	29	24
16	0	11	20	27	32	37	39	40	40	40	38	36	33	29	24
17	0	11	20	28	33	37	39	40	41	40	38	36	33	29	24
18	0	11	21	28	34	38	40	41	41	40	38	36	33	29	24
19	0	11	21	29	34	38	40	41	41	40	38	36	33	29	24
20	0	12	22	29	35	39	41	41	41	40	38	36	33	29	24
21	0	12	22	30	36	39	41	42	42	40	39	36	33	29	24
22	0	12	23	30	36	40	42	42	42	41	39	36	33	29	24
23	0	13	23	31	37	40	42	43	42	41	39	36	33	29	24
24	0	13	24	32	38	41	43	43	42	41	39	36	33	29	24
25	0	14	25	33	38	42	43	43	43	41	39	36	33	29	24
26	0	14	26	34	39	42	44	44	43	42	39	36	33	29	24
27	0	14	26	35	40	43	44	44	43	42	39	36	33	29	24
28	0	15	27	35	41	43	45	45	44	42	40	37	33	29	24
29	0	15	28	36	41	44	45	45	44	42	40	37	33	29	24

By subtracting the parallactic angle, for the respective times of beginning, greatest phase, and ending, from the foregoing angles of position of the moon's centre from the north towards the east, we shall evidently obtain the same angles from the zenith or vertex towards the east.

If, however, the operation be repeated for the accurate determination of the times of beginning and ending, we shall have in the calculations the angle M also at these times. Let ι_1, ω_1, M_1 be the angles appertaining to the beginning, and ι_2, ω_2, M_2 those for the ending, and we shall evidently have the following values, which will be more accurate than the preceding:

Parallactic Angles for the Latitude of Greenwich.

(*same sign as h*)

Arguments: Apparent Hour Angle and *Declination.*

Dec. South.	Hour Angle *h*.														
	° 0	° 10	° 20	° 30	° 40	° 50	° 60	° 70	° 80	° 90	° 100	° 110	° 120	° 130	° 140
°	°	°	°	°	°	°	°	°	°	°	°	°	°	°	°
0	0	8	15	22	27	31	35	37	38	39	38	37	35	31	27
1	0	8	15	21	27	31	34	37	38	39	38	37	35	32	27
2	0	8	15	21	27	31	34	37	38	39	38	37	35	32	28
3	0	8	15	21	26	31	34	36	38	39	38	37	35	32	28
4	0	7	15	21	26	31	34	36	38	39	38	37	35	32	28
5	0	7	15	21	26	30	34	36	38	39	39	38	36	33	28
6	0	7	14	20	26	30	34	36	38	39	39	38	36	33	29
7	0	7	14	20	26	30	34	36	38	39	39	38	36	33	29
8	0	7	14	20	25	30	33	36	38	39	39	38	36	34	29
9	0	7	14	20	25	30	33	36	38	39	39	38	37	34	30
10	0	7	14	20	25	30	33	36	38	39	39	39	37	34	30
11	0	7	14	20	25	29	33	36	38	39	39	39	37	35	31
12	0	7	14	20	25	29	33	36	38	39	40	39	38	35	31
13	0	7	14	19	25	29	33	36	38	39	40	39	38	35	31
14	0	7	13	19	25	29	33	36	38	39	40	40	38	36	32
15	0	7	13	19	24	29	33	36	38	39	40	40	39	36	32
16	0	7	13	19	24	29	33	36	38	40	40	40	39	37	32
17	0	7	13	19	24	29	33	36	38	40	41	40	39	37	33
18	0	7	13	19	24	29	33	36	38	40	41	41	40	38	34
19	0	7	13	19	24	29	33	36	38	40	41	41	40	38	34
20	0	7	13	19	24	29	33	36	38	40	41	41	41	39	35
21	0	6	13	19	24	29	33	36	39	40	42	42	41	39	36
22	0	6	13	19	24	29	33	36	39	41	42	42	42	40	36
23	0	6	13	18	24	29	33	36	39	41	42	43	42	40	37
24	0	6	13	18	24	29	33	36	39	41	42	43	43	41	38
25	0	6	13	18	24	29	33	36	39	41	43	43	43	42	38
26	0	6	13	18	24	29	33	36	39	42	43	44	44	42	39
27	0	6	13	18	24	29	33	36	39	42	43	44	44	43	40
28	0	6	12	18	24	29	33	37	40	42	44	45	45	43	41
29	0	6	12	18	24	29	33	37	40	42	44	45	45	44	41

$$\text{For} \left\{\begin{array}{c}\text{beginning}\\ \text{greatest phase}\\ \text{ending}\end{array}\right\} \angle \text{ of } ☽\text{'s centre from N. towards E.} = \left\{\begin{array}{l}(-\iota_1)-\omega_1\\ (-\iota)\\ (-\iota_2)+\omega_2\end{array}\right\}$$

$$\angle \text{ of } ☽\text{'s centre from vertex towards E.} = \left\{\begin{array}{l}(-\iota_1)-\omega_1-M_1\\ (-\iota)-M\\ (-\iota_2)+\omega_2-M_2\end{array}\right\} \qquad (20)$$

These angles relate to the natural appearance or direct images of the bodies. For the same angles, as they will appear through an inverting telescope, ± 180° must be applied: this may be simply done by using (180° — ι) instead of (— ι).

To find the time when the apparent conjunction takes place, let t denote the interval, in units of an hour, to be applied to the time of the true conjunction, and h the common hour angle of the bodies at the true conjunction. Then the position of the sun, not being supposed to be influenced by parallax, the common apparent hour angle of the bodies, at the time of the apparent conjunction, will be $h' = h + 15^\circ . t$; and therefore at this time,

$$a = a_1 t, \qquad \triangle a = \left(P' \frac{\cos l}{\cos D}\right) \sin (h + 15^\circ . t),$$

so that the conditi n for apparent conjunction, viz. $a' = a - \triangle a = 0$, gives

$$a_1 t - \left(P' \frac{\cos l}{\cos D}\right) \sin (h + 15^\circ . t) = 0 \quad . \quad . \quad . \quad . \quad . \quad (21)$$

for the determination of the interval t, which from this equation will be best found perhaps, by the usual method of double position. We only want, however, an approximate value, and may therefore avoid much unnecessary labor in estimating this time. Thus, at the time of true conjunction, the same approximate formulæ may be adopted as used at page 385, viz.:

$$\triangle a = P' \frac{\cos l}{\cos D} \sin h,$$

$$\triangle a_1 = P' \left(\frac{dh}{dt} \sin 1''\right) \frac{\cos l}{\cos D} \cos h,$$

in which $\frac{dh}{dt}$ applies to the moon. It is evident, then, as the true positions of the bodies have no difference of right ascension, that $\triangle a$ is the apparent difference of right ascension; and consequently, as the relative apparent motion in right ascension is $a_1 - \triangle a_1$ or $a_1 - P' \left(\frac{dh}{dt} \sin 1''\right) \frac{\cos l}{\cos D} \cos h$, the correction t to be applied to the time of true conjunction to get that of the apparent, will be

$$t = \frac{P' \frac{\cos l}{\cos D} \sin h}{a_1 - P' \left(\frac{dh}{dt} \sin 1''\right) \frac{\cos l}{\cos D} \cos h} = \frac{\sin h}{a_1 \frac{\cos D}{P' \cos l} - \left(\frac{dh}{dt} \sin 1''\right) \cos h}$$

To facilitate the calculation of this expression, we may use 57′ as a mean value for P' and 14° as a mean value of D. Assume, therefore,

$$\left.\begin{aligned} f &= \frac{100 \cos D}{P' \cos l} = \frac{100}{57} \cdot \frac{\cos 14^\circ}{\cos l} = \frac{[0.23103]}{\cos l} \\ 6 &= 100 \left(\frac{dh}{dt} \sin 1''\right) \cos h = [1.40274] \cos h \\ 6^{(1)} &= 100 \sin h \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (22)$$

for which the nearest whole numbers will suffice, and we shall have

$$t = \frac{6^{(1)}}{a_1 . f - 6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The values of the factor f are given for various principal places in the table at page 406: for any place not contained in that table it can be computed from the above expression, and used as a constant factor for all eclipses at that place. The values of 6, $6^{(1)}$, are also tabulated at page 405, where, for convenience, the argument h is given in time.

II.—Formulæ of Reduction to different Places

Before quitting this subject we shall give a method of calculating numerical equations which will serve to determine, with much ease and with sufficient accuracy, the circumstances of an eclipse of the sun for any place comprised within a certain range of country. To effect this purpose in the most ample manner, in again proceeding with the general determination of the time of a phase, whose apparent distance of centres is Δ', we shall, in the expressions, separate as much as possible the quantities which involve the position of the place on the earth. The values of the co-ordinates x, y, given at p. 387, observing that $a - \Delta a = a'$, may be put down as follows:

$$x = [(D + a' \text{ corr.}) - \delta] - \Delta D,$$
$$y = a \cos D' - \Delta a \cos D',$$

and will thus consist of two terms, over the former of which the particular place on the earth has but little influence. If ι denote, as before, the inclination of the apparent relative orbit, these ordinates resolved in the direction of n, perpendicular to the orbit, and in the direction of the orbit, will give $x \cos \iota - y \sin \iota$, and $x \sin \iota + y \cos \iota$. It is evident, then, that $x \cos \iota - y \sin \iota$ represents n, the nearest approach, and $x \sin \iota + y \cos \iota$ the distance of the moon from it, which distance is estimated in the direction of her motion. At the time of the beginning or ending of the phase, the distance of the moon past the nearest approach, or greatest phase, will be $\mp \Delta' \sin \omega$; therefore the moon precedes this position by a distance equal to $\mp \Delta' \sin \omega - (x \sin \iota + y \cos \iota)$, which, divided by $\frac{y_1}{\cos \iota}$, the hourly motion in the orbit, gives $\mp \frac{\Delta' \cos \iota}{y_1} \sin \omega - \frac{\cos \iota}{y_1}(x \sin \iota + y \cos \iota)$ for the interval, in units of an hour, to be applied to the assumed time T to get the time t when the phase takes place. Assume, therefore,

$$k = [3.55630] \frac{\Delta' \cos \iota}{y_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and, the time being counted in seconds,

$$t = T \mp k \sin \omega - \frac{k}{\Delta'}(x \sin \iota + y \cos \iota) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also, $x \cos \iota - y \sin \iota$ expressing the nearest approach, we evidently have

$$\cos \omega = \frac{x \cos \iota - y \sin \iota}{\Delta'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Make now the following assumptions:

$$\left.\begin{aligned} p &= \frac{(D + a' \text{ corr.}) - \delta}{\Delta'} \cos \iota - \frac{a \cos D'}{\Delta'} \sin \iota \\ q &= \frac{k}{\Delta'}[(D + a' \text{ corr.}) - \delta] \sin \iota + \frac{k}{\Delta'} a \cos D' \cos \iota \end{aligned}\right\} \quad . \quad . \quad . \quad (4)$$

$$\left.\begin{aligned} \Delta p &= \frac{\Delta D}{\Delta'} \cos \iota - \frac{\Delta a \cos D'}{\Delta'} \sin \iota \\ \Delta q &= \frac{k}{\Delta'} \Delta D \sin \iota + \frac{k}{\Delta'} \Delta a \cos D' \cos \iota \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and, observing the above values of x and y, the equations (2), (3) will become

$$\left.\begin{aligned} \cos \omega &= p - \Delta p, \\ t &= T \mp k \sin \omega - (q - \Delta q) \end{aligned}\right\} \quad . \quad . \quad (6)$$

Let γ, ψ be determined by the equations

$$\left.\begin{aligned} \gamma \cos \psi &= \frac{(D + a' \text{ corr}) - \delta}{\Delta'} \\ \gamma \sin \psi &= \frac{a \cos D'}{\Delta'} \end{aligned}\right\} \quad \ldots \ldots \quad (7)$$

and p, q will take the following values:

$$\left.\begin{aligned} p &= \gamma \cos (\psi + \iota) \\ q &= k \gamma \sin (\psi + \iota) \end{aligned}\right\}. \quad \ldots \ldots \ldots \quad (8)$$

It yet remains to determine the values of Δp, Δq, which depend on the position of the place of observation. Adopting the notation used in the equations (3), (4), (9), (10), pages 379 and 382, we shall have

$$\Delta a = \frac{[5.31439] A}{1 - n} \cdot \frac{\cos l}{\cos D} \sin h,$$

$$\Delta D = \frac{[5.31439] A}{1 - n_1} \left[\sin l \cos D - \cos l \sin D \frac{\cos (h + \frac{1}{2} \Delta a)}{\cos \frac{1}{2} \Delta a} \right].$$

To simplify the expressions, let

$$b = \frac{[5.31439] A}{(1 - n) \Delta'} \cdot \frac{\cos D'}{\cos D},$$

$$c = \frac{[5.31439] A}{(1 - n_1) \Delta'} \cdot \cos D, \qquad a = \frac{[5.31439] A}{(1 - n_1) \Delta'} \cdot \sin D;$$

and

$$\Delta a = \frac{b \Delta' \cos l \sin h}{\cos D'},$$

$$\Delta D = c \Delta' \sin l - a \Delta' \cos l \frac{\cos (h + \frac{1}{2} \Delta a)}{\cos \frac{1}{2} \Delta a},$$

$$= c \Delta' \sin l - a \Delta' \cos l \cos h + a \Delta' \tan \frac{\Delta a}{2} \cos l \sin h.$$

These substituted in (5) give

$$\Delta p = c \cos \iota \sin l - \cos l \left[a \cos \iota \cos h - (a \cos \iota \tan \frac{\Delta a}{2} - b \sin \iota) \sin h \right]$$

$$\Delta q = k c \sin \iota \sin l - \cos l \left[k a \sin \iota \cos h - (k a \sin \iota \tan \frac{\Delta a}{2} + k b \cos \iota) \sin h \right]$$

The value of b contains the factor $\frac{\cos D'}{\cos D}$, for which we have

$$\frac{\cos D'}{\cos D} = \cos \Delta D (1 + \tan D \tan \Delta D).$$

Substitute the first value of $\tan \Delta D$, p. 379, and

$$\frac{\cos D'}{\cos D} = \cos \Delta D \cdot \frac{1 - \rho \sin P \frac{\cos l}{\cos D} \cos (h)}{1 - \rho \sin P [\sin l \sin D + \cos l \cos D \cos (h)]}.$$

Or, putting h instead of (h) in the numerator, which cannot sensibly affect the value of the fraction,

$$\frac{\cos D'}{\cos D} = \cos \Delta D \cdot \frac{1 - n}{1 - n_1}.$$

This, supposing $\cos \Delta D = 1$, reduces the values of the constants a, b, c, to the following:

$$\left. \begin{array}{c} b = \dfrac{[5.31439]\,A}{(1 - n_1)\,\Delta'} \\ c = b \cos D; \qquad a = b \sin D \end{array} \right\} \quad \ldots \ldots \ldots (9)$$

If e be a small arc determined by $g \cos e = b$, $g \sin e = a \tan \frac{\Delta a}{2}$, we shall have

$$a \cos \iota \tan \frac{\Delta a}{2} - b \sin \iota = g \sin(-\iota + e) = g \cos(90° + \iota - e);$$

$$k a \sin \iota \tan \frac{\Delta a}{2} + k b \cos \iota = k g \cos(\iota - e) = k g \sin(90° + \iota - e).$$

However, as e must always be a very small arc, we may suppose $\cos e = 1$ also $g = b$, and, e being expressed in minutes,

$$e = \frac{1}{60} \cdot \frac{a}{b} \cdot \frac{\Delta a}{2} = \frac{a}{120\,b} \cdot \Delta a = [7.9208]\,\Delta a \sin D. \quad \ldots (10)$$

If therefore

$$\chi = (90° + \iota) - e \quad \ldots \ldots \ldots (11)$$

the values of Δp, Δq, will be

$$\left. \begin{array}{l} \Delta p = c \cos \iota \sin l - \cos l\,(a \cos \iota \cos h - b \cos \chi \sin h) \\ \Delta q = k c \sin \iota \sin l - \cos l\,(k a \sin \iota \cos h - k b \sin \chi \sin h) \end{array} \right\} \quad \ldots (12)$$

Assume now

$\lambda =$ the longitude of the place, $+$ east, $-$ west.

$H =$ the true hour angle of the moon, for the meridian of Greenwich.

$$\left. \begin{array}{r} L' = c \cos \iota \\ \gamma' \cos(\psi' - H) = a \cos \iota \\ \gamma' \sin(\psi' - H) = b \cos \chi \end{array} \right\} \quad \ldots \ldots \ldots (13)$$

$$\left. \begin{array}{r} L'' = k c \sin \iota \\ \gamma'' \cos(\psi'' - H) = k a \sin \iota \\ \gamma'' \sin(\psi'' - H) = k b \sin \chi \end{array} \right\} \quad \ldots \ldots \ldots (14)$$

and we shall have

$$\Delta p = L' \sin l - \gamma' \cos l \cos(\psi' + h - H) = L' \sin l - \gamma' \cos l \cos(\psi' + \lambda),$$
$$\Delta q = L'' \sin l - \gamma'' \cos l \cos(\psi'' + h - H) = L'' \sin l - \gamma'' \cos l \cos(\psi'' + \lambda);$$

so that the equations (6) will become

$$\left. \begin{array}{l} \cos \omega = p - L' \sin l + \gamma' \cos l \cos(\psi' + \lambda) \\ t = (T - q) \mp k \sin \omega + L'' \sin l - \gamma'' \cos l \cos(\psi'' + \lambda) \end{array} \right\} \quad \ldots (15)$$

After computing the constants $k, p, q, L', L'', \psi', \psi''$, by means of the equations (1), (7), (8), (9), (10), (11), (13), and (14), we shall thus have two numerical equations for the determination of ω and the Greenwich time t of the phase, for any place whose latitude is l and longitude λ. The accuracy of the determination will principally depend on the proximity of the resulting time t to the assumed time T; and therefore the result will be near the truth for all places where the phase will take place near to this time.

In making these calculations for any particular portion of country, which for the partial phase will be necessary for both the beginning and ending, it will be best in the first instance to fix upon a place near the centre and compute the eclipse for that place, which computation will furnish good mean values for the data D, τ, δ, a' corr., ΔD, Δa, ι, y_1, Δ', A, and comp. log $(1 - n_1)$.

By supposing
$$\left.\begin{array}{ll}\xi'\cos l' = \gamma', & \xi''\cos l'' = \gamma'' \\ \xi'\sin l' = -L', & \xi''\sin l'' = -L''\end{array}\right\} \quad \ldots\ldots (16)$$

the expressions
$$-L'\ \sin l + \gamma'\ \cos l \cos(\psi' + \lambda),$$
$$-L''\sin l + \gamma''\cos l\cos(\psi'' + \lambda),$$

will take the forms
$$\xi'\ [\sin l'\ \sin l + \cos l'\ \cos l\cos(\psi' + \lambda)],$$
$$\xi''[\sin l''\sin l + \cos l''\cos l\cos(\psi'' + \lambda)];$$

and, without the factors ξ', ξ'', will represent the cosines of the distances of the proposed place from two other places whose latitudes are l', l'', and west longitudes ψ', ψ''. The former of these two places will be near to the southern pole of the true relative orbit, and the latter will be near to the orbit itself, and will precede the moon by a distance nearly equal to 90°.

For purposes which do not require great minuteness, the preceding equations will admit of some simplification by neglecting the small angle e. Add the squares of the equations (13) and (14), observing that $c^2 + a^2 = b^2$, and
$$L'^2 + \gamma'^2 = \quad b^2(\cos^2\iota + \cos^2\chi),$$
$$L''^2 + \gamma''^2 = k^2 b^2(\sin^2\iota + \sin^2\chi);$$

which give the general relation
$$L'^2 + \gamma'^2 + \frac{L''^2}{k^2} + \frac{\gamma''^2}{k^2} = 2b^2 \quad \ldots\ldots (17)$$

By neglecting e, $\chi = 90° + \iota$, $\cos\chi = -\sin\iota$, $\sin\chi = \cos\iota$; and then
$$L'^2 + \gamma'^2 = b^2,$$
$$L''^2 + \gamma''^2 = k^2 b^2;$$

which united with the equations (16) give $\xi' = b$, $\xi'' = k\,b$, and hence
$$\sin l' = -\frac{L'}{\xi'} = -\frac{c\cos\iota}{b} = -\cos D\cos\iota;$$
$$\gamma' = \xi'\cos l' = b\cos l';$$
$$\sin(\psi' - H) = \frac{b\cos\chi}{\gamma'} = -\frac{b\sin\iota}{b\cos l'} = -\frac{\sin\iota}{\cos l'};$$
$$\sin l'' = -\frac{L''}{\xi''} = -\frac{k c\sin\iota}{k b} = -\cos D\sin\iota;$$
$$\gamma' = \xi''\cos l'' = k\,b\cos l'';$$
$$\sin(\psi'' - H) = \frac{k\,b\sin\chi}{\gamma''} = \frac{k\,b\cos\iota}{k\,b\cos l''} = \frac{\cos\iota}{\cos l''}.$$

Or
$$\left.\begin{array}{c}\sin l' = -\cos D\cos\iota \\ L' = -b\sin l'; \qquad \gamma' = b\cos l' \\ \sin(\psi' - H) = -\dfrac{\sin\iota}{\cos l'}\end{array}\right\} \quad (18)$$

$$\left.\begin{array}{c}\sin l'' = -\cos D\sin\iota \\ L'' = -k\,b\sin l''; \qquad \gamma'' = k\,b\cos l'' \\ \sin(\psi'' - H) = \dfrac{\cos\iota}{\cos l''}\end{array}\right\} \quad \ldots (19)$$

These may be employed instead of the equations (13) and (14); or the equations (13) and (14) may be adopted in their reduced form, viz.:

$$\left.\begin{aligned} \frac{L'}{b} &= \cos D \cos \iota \\ \frac{\gamma'}{b} \cos (\psi' - H) &= \sin D \cos \iota \\ \frac{\gamma'}{b} \sin (\psi' - H) &= - \sin \iota \end{aligned}\right\} \quad \ldots\ldots \quad (20)$$

$$\left.\begin{aligned} \frac{L''}{k\,b} &= \cos D \sin \iota \\ \frac{\gamma''}{k\,b} \cos (\psi'' - H) &= \sin D \sin \iota \\ \frac{\gamma''}{k\,b} \sin (\psi'' - H) &= \cos \iota \end{aligned}\right\} \quad \ldots\ldots \quad (21)$$

in which the coefficients c, a, will not be required.

III.—Transits of Mercury and Venus over the Disk of the Sun.

These phenomena are, in many respects, analogous to that of an annular eclipse of the sun, and admit of a similar calculation; the principal distinction consists in the negative sign of the relative motion of the planet in right ascension, which will make the inclination of the orbit always obtuse, and therefore render some modifications necessary in the determination of the particular species of the other angles which enter into the computation. To avoid any confusion that might thus arise, we shall adopt the sun as the movable body, and refer his positions to that of the planet which we now suppose to be stationary. Thus,

δ = the ☉'s declination;
D = the planet's declination;
π = the ☉'s equatorial horizontal parallax,
P = the planet's equatorial horizontal parallax;
α = ☉'s right ascension *minus* that of the planet;
$x = (\delta' + \alpha' \text{ corr.}) - D$;
$y = \alpha' \cos \delta'$;
x_1 = the ☉'s motion in declination *minus* that of the planet;
y_1 = (☉'s motion in right ascension *minus* that of planet) . $\cos \delta'$;

and so we might proceed as with an eclipse of the sun, only observing that the relative parallax $\rho\ (\pi - P)$ is a negative quantity, and that the positions of the contacts on the limb of the sun, as in the case of an occultation, will be at points opposite to those which come out in the calculation. However, as the relative parallax is always very small, the ingress and egress of the planet will be seen at all places on the earth at nearly the same absolute time; it will, for this reason, be best to compute first the circumstances for the centre of the earth, and then to ascertain the small variations produced by parallax for any assumed place on the surface, which may be readily deduced from the preceding equations for the reduction of an eclipse of the sun. Let w, (t), be the values of ω, t, for the centre of the earth, and, by separating the effects of parallax from the equations (6),

$$\cos \mathrm{w} = p,$$

$$(t) = (T - q) \mp k \sin \mathrm{w},$$

$$\Delta \cos \mathrm{w} = \Delta\, p, \qquad \Delta\, t = -\Delta\, q \mp k\, \Delta \sin \mathrm{w}.$$

But, as the quantities $\Delta \cos \mathrm{w}$, $\Delta \sin \mathrm{w}$ are very small, $\Delta \sin \mathrm{w} = -\Delta \cos \mathrm{w}\,\frac{\cos \mathrm{w}}{\sin \mathrm{w}}$ that is, $\Delta \sin \mathrm{w} = -\Delta\, p\,\frac{\cos \mathrm{w}}{\sin \mathrm{w}}$. Therefore,

$$\Delta\, t = -\Delta\, q \pm k\, \Delta\, p\,\frac{\cos \mathrm{w}}{\sin \mathrm{w}} = \pm \left(k\, \Delta\, p\,\frac{\cos \mathrm{w}}{\sin \mathrm{w}} \mp \Delta\, q\right).$$

In this expression substitute the values of $\Delta\, p$, $\Delta\, q$, according to the equations (12), and we find $\Delta\, t =$

$$\pm\left[k\,c\,\frac{\cos[-\iota \mp \mathrm{w}]}{\sin \mathrm{w}} \sin l - \cos l\left(k\,a\,\frac{\cos[-\iota \mp \mathrm{w}]}{\sin \mathrm{w}} \cos h - k\,b\,\frac{\cos[-\chi \mp \mathrm{w}]}{\sin \mathrm{w}} \sin h\right)\right],$$

in which $b = \frac{\rho\,(\pi - P)}{\Delta} = -\frac{\rho\,(P - \pi)}{\Delta}$, $c = b \cos \delta$ and $a = b \sin \delta$.

Because of the smallness of the parallax, the angle e will not be appreciable, and consequently $\chi = 90^\circ + \iota$, $\cos\,[-\chi \mp \mathrm{w}] = \sin\,[-\iota \mp \mathrm{w}]$. We shall therefore have for the time of ingress or egress the following general expression, in which the terms within the brackets depend on the position of the place of observation; also the upper signs apply to the ingress, and the under signs to the egress.

$$t = T - q \mp k \sin \mathrm{w}$$

$$\mp k\,b\left[\cos \delta\,\frac{\cos[-\iota \mp \mathrm{w}]}{\sin \mathrm{w}} \sin l - \left(\sin \delta\,\frac{\cos[-\iota \mp \mathrm{w}]}{\sin \mathrm{w}} \cos h - \frac{\sin[-\iota \mp \mathrm{w}]}{\sin \mathrm{w}} \sin h\right) \cos l\right]$$

Assuming $k'' = \frac{-k\,b}{\rho \sin \mathrm{w}}$, this expression will resolve into the following:

$$\left.\begin{aligned} \tan \iota &= \frac{x_1}{y_1} \\ k &= [3.55630]\,\frac{\Delta \cos \iota}{y_1} \end{aligned}\right\} \quad \ldots\ldots \quad (a)$$

$$\left.\begin{aligned} \gamma \cos \psi &= \frac{(\delta + a \text{ corr.}) - D}{\Delta} \\ \gamma \sin \psi &= \frac{a \cos \delta}{\Delta} \end{aligned}\right\} \quad \ldots\ldots \quad (b)$$

$$\left.\begin{aligned} \cos \mathrm{w} &= \gamma \cos(\psi + \iota) \\ q &= k\,\gamma \sin(\psi + \iota) \\ (t) &= T - q \mp k \sin \mathrm{w} \end{aligned}\right\} \quad \ldots\ldots \quad (c)$$

$$\left.\begin{aligned} k'' &= k\,.\,\frac{(P - \pi)}{\Delta \sin \mathrm{w}} \\ \frac{L''}{k''} &= \cos\,[(-\iota) \mp \mathrm{w}] \cos \delta \\ \frac{\gamma''}{k''} \cos(\psi'' - H) &= \cos\,[(-\iota) \mp \mathrm{w}] \sin \delta \\ \frac{\gamma''}{k''} \sin(\ '' - H &= \sin\,[(-\iota) \mp \mathrm{w}] \end{aligned}\right\} \quad \ldots\ldots \quad (d)$$

$$t = (t) \mp [\gamma''\,\rho \cos\ \ \cos(\psi'' + \lambda) - L''\,\rho \sin\ \ ] \quad \ldots\ldots\ldots \quad (e)$$

In these equations,

$H =$ the ☉'s true hour angle from the meridian of Greenwich, at the time (t).

$$\text{For} \left\{\begin{matrix}\text{exterior}\\ \text{interior}\end{matrix}\right\} \text{contact of limbs, } \triangle = \left\{\begin{matrix}\sigma + s\\ \sigma - s\end{matrix}\right\}$$

$$\text{For contact of centre of planet with ☉'s limb, } \triangle = \sigma;$$

s denoting the true semi-diameter of the planet, and σ that of the sun.

The equations (a), (b), (c), (d) will serve to determine the constants (t), γ'', L'', ψ'', for the times of ingress and egress, and then there will result two numerical equations of the form (e) to reduce the phenomena to any place on the earth's surface.

For the points on the limb of the sun, we shall have

$$\text{At} \left\{\begin{matrix}\text{ingress}\\ \text{egress}\end{matrix}\right\}, \text{angle from N. towards E.} = \left\{\begin{matrix}(180^\circ - \iota) - \text{w}\\ (180^\circ - \iota) + \text{w}\end{matrix}\right\} \text{for } \textit{direct} \text{ image.}$$

$$\text{or} \left\{\begin{matrix}(-\iota) - \text{w}\\ (-\iota) + \text{w}\end{matrix}\right\} \text{for } \textit{inverted} \text{ image.}$$

which will be sufficiently accurate for all places on the earth.

The time T may be assumed near to the time of conjunction in longitude, or right ascension, as it may suit convenience. For Mercury, if very minute accuracy is wanted, it may be necessary, for more correct values of (t), to assume two times T near to the times of ingress and egress; but it is very questionable whether such a precarious extent of accuracy would sufficiently recompense the time expended on the calculation.

IV.—Occultations of Stars by the Moon.

These may be calculated in the same manner as eclipses of the sun, the only difference in the operation consisting in the star having neither motion, parallax, nor semi-diameter. But where great minuteness is not wanted, these particular circumstances will afford some degree of simplification to the expressions, if that parallax of the moon be adopted which would answer to the star as an apparent place, since this parallax, at the times of immersion and emersion, will then be precisely that of the respective points of the moon's limb which come in contact with the star; and thus the augmentation of the moon's semi-diameter will be evaded, so that the true semi-diameter may be employed. For this novel and judicious expedient we are indebted to Carlini.—See *Zach's Correspondance*, vol. xviii., page 528.

As in the case of the sun, let δ denote the declination, and h the hour angle of the star, and let P represent the equatorial horizontal parallax of the moon. Then, for the effects of parallax in right ascension and declination, we must substitute δ for D', and h for h in the formulæ (2) at p. 379, which thus become, disregarding $\frac{1}{2} \triangle h$,

$$\triangle a = \rho P \frac{\cos l}{\cos D} \sin h,$$

$$\triangle D = \rho P (\sin l \cos \delta - \cos l \sin \delta \cos h).$$

As soon as the immersion takes place, these expressions will represent the parallax of that point of the moon's limb which is in contact with the star; and therefore the application of this parallax to the centre of the moon will produce an apparent distance $\triangle'$ of the centres, equal to the *true* semi-diameter s of the moon. Also as the star, in the course of the occultation, is only affected with its apparent diurnal motion, the hourly variations of the above values will be

$$\Delta\, a_1 = \rho\, P \left(\frac{dh}{dt} \sin 1''\right) \frac{\cos l}{\cos D} \cos h.$$

$$\Delta\, D_1 = \rho\, P \left(\frac{dh}{dt} \sin 1''\right) \cos l \sin \delta \sin h;$$

in which $\frac{dh}{dt}$ is 15° 2′ 28″, the hourly diurnal motion of the earth, and therefore $\frac{dh}{dt} \sin 1'' = [9.41916]$.

Assume

$$\left.\begin{aligned} \phi^{(1)} &= \rho \cos l = \frac{\cos l'}{\sqrt{1 - e^2 \sin^2 l'}} \\ \phi^{(2)} &= \rho \sin l = \frac{(1 - e^2) \sin l'}{\sqrt{1 - e^2 \sin^2 l'}} \\ \phi^{(3)} &= \rho \cos l \frac{dh}{dt} \sin 1'' = [9.41916]\, \phi^{(1)} \end{aligned}\right\} \quad \ldots\ldots (1)$$

which are constant coefficients depending on the latitude of the place; then

$$\Delta\, a = \frac{\phi^{(1)} . P}{\cos D} \sin h, \qquad \Delta\, a_1 = \frac{\phi^{(3)} . P}{\cos D} \cos h.$$

$$\Delta\, D = (\phi^{(2)} \cos \delta - \phi^{(1)} \sin \delta \cos h) . P, \qquad \Delta\, D_1 = \phi^{(3)} . P \sin \delta \sin h.$$

If, in the values of $\Delta\, a$, $\Delta\, a_1$, we use $\cos \delta$ instead of $\cos D$, the values of x, y, x_1, y_1, p. 387, will become

$$\left.\begin{aligned} x &= (D - \delta) - (\phi^{(2)} . P \cos \delta - \phi^{(1)} . P \sin \delta \cos h) \\ y &= a \cos \delta \quad - \phi^{(1)} . P \sin h \\ x_1 &= D_1 \quad - \phi^{(3)} . P \sin \delta \sin h \\ y_1 &= a_1 \cos \delta \quad - \phi^{(3)} . P \cos h \end{aligned}\right\} \quad \ldots\ldots (2)$$

in which we have disregarded the a correction.

With the values of x, y, x_1, y_1, so found, we may then proceed with the equations (16) and (18), pages 387 and 388, as in the case of a solar eclipse.

This method is similar, and, as far as accuracy goes, the same as the recent method of Professor Bessel, who divides all the quantities by the equatorial horizontal parallax of the moon. He assumes

$$\left.\begin{aligned} p &= \frac{a \cos \delta}{P}, \qquad & p' &= \frac{a_1 \cos \delta}{P} \\ q &= \frac{D - \delta}{P}, & q' &= \frac{D_1}{P} \end{aligned}\right\} \quad \ldots\ldots (3)$$

$$\left.\begin{aligned} u &= \phi^{(1)} \sin h, & u' &= \phi^{(3)} \cos h \\ v &= \phi^{(2)} \cos \delta - \phi^{(1)} \sin \delta \cos h, \qquad & v' &= \phi^{(3)} \sin \delta \sin h \end{aligned}\right\} \quad \ldots (4)$$

so that if we change the signification of the symbols x, y, x_1, y_1, and suppose them now to represent the preceding values divided by P, we shall have

$$\left.\begin{aligned} x &= q - v, \qquad & x_1 &= q' - v' \\ y &= p - u, & y_1 &= p' - u' \end{aligned}\right\} \quad \ldots\ldots (5)$$

These values being adopted, in proceeding with the equations (16) and (18) we must use $\Delta' = \frac{s}{P}$, the value of which, according to Burckhardt's *Tables de la Lune* (Paris, 1812), p. 73, is [9.43537]. Much facility is thus given to the calculation of occultations, for different places, if the values of p, q, p', q', which are indepen-

dent of geographical position, are published; but if these quantities are to be prepared by the computer, the equations (2) will be more simple and advantageous.

The chief difficulty in the calculation of occultations, for any particular place, rests in the selection of the list of stars: in the course of any year a great number will be liable to occultation on the earth generally, though the majority of them will not be occulted at the particular place for which the special calculations are to be made. It will therefore be expedient to reject such stars as may at different stages of the calculation be shown to violate any conditions necessary for the existence of the occultation, its appearance above the horizon, or its exemption from the glare of sun-light. For the general list we may observe, that the difference of declination at the time of conjunction must be within the limit of about 1° 30′, and that all stars, whose conjunctions with the moon occur within two days of new moon, may be omitted. In the process of exclusion for the particular place, the first and most palpable condition is, that at the time of conjunction the sun must be below, or near to, the horizon; if more than half an hour above the horizon, the occultation will surely be useless; another condition is, that the star must be above the horizon; and, to satisfy this, the hour angles at the times of immersion and emersion must be less than its semi-diurnal arc. The value of the hour angle at the time of apparent conjunction may be determined by increasing that at the time of true conjunction by the quantity $\frac{\theta^{(1)}}{a_1 \cdot f - \theta}$, according to the tables on pages 401 and 402; and it may be observed that this hour angle must not exceed the semi-diurnal arc by more than half an hour. For the latitude of Greenwich, the semi-diurnal arcs, allowing 33′ for refraction in the horizon, are shown in the annexed table.

Dec. of Star.	Semi-diurnal Arcs, for the Latitude of Greenwich.			
	Dec. North.		*Dec. South.*	
°	h. m.		h. m.	
0	6 4	+5	6 4	−5
1	6 9	5	5 59	5
2	6 14	5	5 54	5
3	6 19	5	5 49	6
4	6 24	5	5 43	5
5	6 29	5	5 38	5
6	6 34	5	5 33	5
7	6 39	5	5 28	5
8	6 44	6	5 23	5
9	6 50	5	5 18	5
10	6 55	5	5 13	6
11	7 0	6	5 7	5
12	7 6	5	5 2	6
13	7 11	6	4 56	5
14	7 17	6	4 51	6
15	7 23	5	4 45	5
16	7 28	6	4 40	6
17	7 34	6	4 34	6
18	7 40	7	4 28	6
19	7 47	6	4 22	7
20	7 53	7	4 15	6
21	8 0	6	4 9	7
22	8 6	7	4 2	6
23	8 13	8	3 56	7
24	8 21	7	3 49	8
25	8 28	8	3 41	7
26	8 36	8	3 34	8
27	8 44	9	3 26	8
28	8 53	9	3 18	9
29	9 2	+10	3 9	−9
30	9 12		3 0	

As a final test for the exclusion of unnecessary stars, it is useful to calculate the extreme limits of latitude between which the star will be visibly occulted on the earth. These will evidently appertain to the extreme northern and southern points of the northern and southern limits of contact, determined as for a solar eclipse. a point in the northern or southern limit will depend on the formulæ Nos. 27, 28, pages 359–60. Thus,

$$\cos w = \frac{n \pm \Delta'}{P'},$$

$$\sin Z = \frac{\cos w}{\cos \omega'}, \qquad M = -\iota \pm \omega';$$

and thence,

$$\sin l = \sin D' \cos Z + \cos D' \sin Z \cos M.$$

It is now our object to ascertain what value of ω' will render the value of l, so deduced, a maximum or a minimum, and what will be the corresponding value of l. Let ϕ be an arc determined by the equation,

$$\cos Z = \cos\phi \sin \mathrm{w} \quad \ldots \ldots \ldots \quad (6)$$

Then by uniting with it the equation

$$\cos\omega' \sin Z = \cos \mathrm{w} \quad \ldots \ldots \quad (7)$$

we infer that

$$\sin\omega' \sin Z = \sin\phi \sin \mathrm{w} \quad \ldots \ldots \quad (8)$$

because the squares of these three equations added together will give *unity* on each side. By these equations we shall hence have

$$\begin{aligned} \sin D' \cos Z &= \sin D' \cos\phi \sin \mathrm{w}, \\ \sin Z \cos M &= \sin Z (\cos\iota \cos\omega' \mp \sin\iota \sin\omega'), \\ &= (\cos\omega' \sin Z) \cos\iota \mp (\sin\omega' \sin Z) \sin\iota, \\ &= \cos\iota \cos \mathrm{w} \mp \sin\iota \sin\phi \sin \mathrm{w}; \end{aligned}$$

and, consequently,

$$\sin l = \cos D' \cos\iota \cos \mathrm{w} + \sin \mathrm{w} (\sin D' \cos\phi \mp \cos D' \sin\iota \sin\phi),$$

which now involves only one variable ϕ. Again, assume two arcs, θ, ψ, which will fulfil the equations,

$$\cos\theta \cos\psi = \sin D' \quad \ldots \ldots \ldots \quad (9)$$

$$\cos\theta \sin\psi = \pm \cos D' \sin\iota \quad \ldots \ldots \ldots \quad (10)$$

A third equation will follow from these, viz.:

$$\sin\theta = \cos D' \cos\iota \quad \ldots \ldots \ldots \quad (11)$$

because, as before, the squares of these three equations will together make *unity*. The value of sin l will now become

$$\sin l = \cos \mathrm{w} \sin\theta + \sin \mathrm{w} \cos\theta \cos(\phi + \psi).$$

The angle $\phi + \psi$ being the only variable in this expression, it is evident that the greatest value of l will have $\phi + \psi = 0$, and the least $\phi + \psi = 180°$. Therefore,

$$\left.\begin{matrix}\text{greatest} \\ \text{least}\end{matrix}\right\} \text{value of } l = \left\{\begin{matrix}\theta + \mathrm{w} \\ \theta - \mathrm{w}\end{matrix}\right\}, \text{ using w for } \left\{\begin{matrix}\text{northern} \\ \text{southern}\end{matrix}\right\} \text{limit.}$$

These would be the extreme latitudes for the appearance of the occultation if the earth were a transparent body; as this, however, is not the case, it will be necessary that the star should be above the horizon, a condition not included in the preceding equations. The zenith distance Z must not exceed 90°, and therefore cos Z must necessarily be a positive quantity.

By the equation (6) cos Z must have the same sign as cos ϕ, and this must be the same as + cos ψ for northern limit, or — cos ψ for southern limit, because in the former case $\phi + \psi = 0$, and in the latter $\phi + \psi = 180°$. But, by (9), cos ψ must have the same sign as D'. Consequently,

$$\text{For} \left\{\begin{matrix}\text{northern} \\ \text{southern}\end{matrix}\right\} \text{limit, } \cos Z \text{ has the same sign as } \left\{\begin{matrix}+D', \\ -D'.\end{matrix}\right.$$

It is evident, therefore, that the extreme northern limit will have the star below the horizon, and be excluded when D' is negative, and that for the same reason the southern limit will be excluded when D' is positive. Thus the only admissible extreme limit will be determined by the equations

$$\cos \mathrm{w} = \frac{n \pm \Delta'}{P'}, \qquad l_1 = \theta \pm \mathrm{w} \quad \ldots \quad (12)$$

using upper signs when D' is positive, and under signs when D' is negative.

The other limit for the actual appearance of the occultation will evidently be one

of the two places where the other limiting line meets the rising and setting limits, and will be determined by

$$\cos w = \frac{n \mp \Delta'}{P'}, \qquad \sin l_2 = \cos D' \cos\left[(-\iota) \mp w\right] \quad . \quad . \quad . \quad . \quad (13)$$

using, as before, upper signs when D' is positive, and under signs when D' is negative.

The equations (11), (12), (13), for convenience in determining the species of the angles, may be put in the following form:

$$\left.\begin{aligned} &\cos w_1 = \frac{\mp n - \Delta'}{P'}, \qquad \cos w_2 = \frac{\mp n + \Delta'}{P'} \\ &\sin\theta = \cos D' \cos\iota \\ &l_1 = w_1 - \theta \\ &\sin l_2 = \mp \cos D' \cos(w_2 - \iota) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (14)$$

observing that w_1, w_2, θ, and ι, must here take the same sign as D'; also,

$$\left.\begin{matrix}\text{upper}\\ \text{under}\end{matrix}\right\}\text{signs when } D' \text{ is}\left\{\begin{matrix}\text{positive,}\\ \text{negative.}\end{matrix}\right.$$

These formulæ are applicable to a solar eclipse. For an occultation of a star by the moon, P' will be the moon's horizontal parallax, and Δ' her semidiameter, which, as these limits are not wanted very accurately, may be regarded as true quantities; also, we may neglect u and so take δ instead of D'. Since $\frac{s}{P} = [9.43537] = .2725$, the formulæ for an occultation will hence be

$$\left.\begin{aligned} &\tan\iota = \frac{D_1}{a_1 \cos\delta} \qquad n = (\text{diff. dec.}) \cos\iota \\ &\cos w_1 = \mp \frac{n}{P} - .2725, \qquad \cos w_2 = \mp \frac{n}{P} + .2725 \\ &\sin\theta = \cos\delta \cos\iota \\ &l_1 = w_1 - \theta, \qquad \sin l_2 = \mp \cos\delta \cos(w_2 - \iota) \end{aligned}\right\} \quad . \quad . \quad . \quad (15)$$

in which we also give to the angles w_1, w_2, ι, θ, the same sign as δ, and use upper signs when δ is positive, and under signs when δ is negative. We may also observe, that,

1. When δ is north, l_1 is the most northern limit; and when δ is south, l_1 is the most southern limit.

2. When w_1 is imaginary, l_1 will be 90°, and of the same name as δ. In this case the occultation will be visible about the pole of the earth, which is presented to the star; the visibility will extend beyond the extremity of the disk of the earth as it would be seen from the star.

3. When w_2 is imaginary, l_2 will be the complement of δ, and of a different name from δ. In this case, if we consider the disk of the earth as seen from the star, the visibility of the occultation will extend beyond that extremity of the disk which has the pole on the other side of it.

After an occultation is computed for any particular place, if we deduct the star's right ascension from the sidereal times of immersion and emersion we shall get the hour angles of the star, + West, — East. By comparing these hour angles with the semi-diurnal arc of the star, we can distinctly ascertain the positions of the star with respect to the horizon.

V.—Eclipses of the Moon by the Earth's Shadow.

These may be also resolved in the same way as those of the sun. The absolute positions of the moon and shadow being independent of the position of the spectator on the earth, the determination of parallaxes will be here unnecessary, which much simplifies the calculation of these eclipses. The considerations requisite to be attended to, by way of distinction, are the following:

$$\text{Semi-diameter of the shadow} = \frac{61}{60}(P' + \pi - \sigma).$$

$$\text{Semi-diameter of the penumbra} = \frac{61}{60}(P' + \pi - \sigma) + 2\,\sigma.$$

Right ascension of centre of shadow = that of the sun $\pm\ 12^h$.

Declination of centre of shadow = that of the sun *with a contrary name.*

The figure of the earth being spheroidal, that of the shadow will deviate a little from a circle, so that, to have a mean radius, the horizontal parallax of the moon must be reduced to a mean latitude of 45°. This will give

$$P' = [9.99929]\ P:$$

P denoting the moon's equatorial horizontal parallax.

Also,

a = right ascens. moon *minus* right ascens. centre of shadow;
x = (dec. moon + a corr.) *minus* dec. centre of shadow;
$y = a \cos D$.

With these we compute according to the equations (16) and (18), pages 387 and 388, observing the following values of Δ':

For $\left\{\begin{matrix}\text{external}\\ \text{internal}\end{matrix}\right\}$ contact with shadow, $\Delta' =$ semi-diam. shadow $\pm\ s$.

For $\left\{\begin{matrix}\text{external}\\ \text{internal}\end{matrix}\right\}$ contact with penumbra, $\Delta' =$ semi-diam. penumbra $\pm\ s$.

The angular positions of the points where the contacts take place will be estimated on the circumference of the shadow or penumbra the same as they were before on the limb of the sun. These angles will therefore be in a reversed position on the disk of the moon, and consequently as they come out from the computation will have reference in the first instance to the inverted appearance of the phase.

The relative orbit of the moon, not being affected with parallax, will not sensibly deviate from a great circle in the course of the eclipse; and hence the assumption of the particular time, on which to found the calculation, will be but of little importance: any convenient time may be assumed near the time of opposition.

It will be unnecessary to add any further remarks. We shall conclude this paper with a tabular recapitulation of the formulæ which relate to the phenomena for a particular place, in which eclipses of the moon, for the sake of clearness, are given separately. The object of this table, like the former one for the general eclipse, is to simplify and expedite, by an easy reference, the actual operations of the computer.

I. ECLIPSE OF THE SUN FOR A PARTICULAR PLACE.

1. h = apparent time of true ☌ in R. A. to nearest minute.

With this as an argument, take out the numbers $\mathfrak{b}$, $\mathfrak{b}^{(1)}$, from the following table:

Table for reducing the *true* to the *app.* ☌ in R. A.

Hour Angle h at true ☌.		$\mathfrak{b}$ + —	$\mathfrak{b}^{(1)}$ same sign as h.	Hour Angle h at true ☌.		$\mathfrak{b}$ + —	$\mathfrak{b}^{(1)}$ same sign as h.
h. m.	h. m.			h. m.	h. m.		
0 0	12 0	25	0	3 0	9 0	18	71
10	11 50	25	4	10	8 50	17	74
20	40	25	9	20	40	16	77
30	30	25	13	30	30	15	79
40	20	25	17	40	20	14	82
50	10	25	22	50	10	14	84
1 0	11 0	24	26	4 0	8 0	13	87
10	10 50	24	30	10	7 50	12	89
20	40	24	34	20	40	11	91
30	30	23	38	30	30	10	92
40	20	23	42	40	20	9	94
50	10	22	46	50	10	8	95
2 0	10 0	22	50	5 0	7 0	7	97
10	9 50	21	54	10	6 50	5	98
20	40	21	57	20	40	4	98
30	30	20	61	30	30	3	99
40	20	19	64	40	20	2	100
50	10	19	68	50	10	1	100
3 0	9 0	18	71	6 0	6 0	0	100

Then, T denoting the approximate mean time of app. ☌, in units of an hour,

$$T = \text{mean time true ☌} + \frac{\mathfrak{b}^{(1)}}{a_1 . f - \mathfrak{b}},$$

in which a_1 must be used in minutes of arc; also $f = \frac{[0.2310]}{\cos l}$, is a factor depending on the latitude, which, for several principal observatories, is, for convenience included in the following table:

Auxiliary Quantities depending on Geographical Position.

Place.	ρ	cot l	cos l	f	Longitude. h. m. s.
Aberdeen . . .	9·99900	+9·81289	9·73637	3·12	W. 0 8 23
Altona	9·99908	+9·87133	9·77576	2·85	E. 0 39 47
Berlin	9·99910	+9·88751	9·78603	2·79	E. 0 53 36
Bedford	9·99911	+9·89345	9·78974	2·76	W. 0 1 52
Cambridge . . .	9·99911	+9·89231	9·78903	2·77	E. 0 0 24
Cape of Good Hope	9·99956	−0·17494	9·91980	2·05	E. 1 13 55
Dublin	9·99909	+9·87385	9·77737	2·84	W. 0 25 22
Edinburgh . . .	9·99902	+9·83256	9·75001	3·03	W. 0 12 44
Greenwich . . .	9·99913	+9·90381	9·79610	2·72	0 0 0
Ormskirk . . .	9·99908	+9·87092	9·77549	2·85	W. 0 11 36
Oxford	9·99912	+9·89939	9·79340	2·74	W. 0 5 2
Kensington . . .	9·99913	+9·90340	9·79586	2·72	W. 0 0 47
Milan	9·99928	+9·99577	9·84736	2·42	E. 0 36 47
Paris	9·99920	+9·94451	9·81997	2·58	E. 0 9 22
Slough	9·99913	+9·90337	9·79584	2·72	W. 0 2 24

2. The time T being computed to the nearest minute, take out the corresponding values of P, π, α, δ, from the Ephemeris; and prepare the constants

$$c = [4.68555]\,\rho,$$

$$A = c\,(P - \pi), \qquad m = A \cos l,$$

$$Q_1 = [4.7172], \qquad Q_2 = m\,Q_1 \sin \delta,$$

$$s = [9.43537]\,P.$$

3. Take out D, δ, a, D_1, a_1, for the time T.

h = sidereal time at place *minus* ☽'s right ascension, to the tenth of a minute, *in arc*.

$$k = \frac{m}{\cos D}, \qquad n = k \cos h,$$

$$\triangle a = [5.31439]\,k \sin h\ [\text{corr. for } n],$$

$$\triangle a_1 = Q_1\,n, \qquad \triangle D_1 = Q_2 \sin h.$$

Correction for n to be taken from the table on page 383.

4.
$$(h) = h + \tfrac{1}{2} \triangle a,$$

$$\tan \theta = \cos (h) \cot l, \qquad G = \cos (h) \cos l,$$

$$\tan M = \frac{\sin \theta}{\cos (\theta + D)} \tan (h), \qquad \tan \epsilon = \tan (\theta + D) \cos M,$$

$$B = \cos M \cos \epsilon;$$

$$\text{check} \ldots \frac{\sin \theta}{\cos (\theta + D)} = \frac{G}{B}.$$

M to be in the same semicircle with h.

$n_1 = A \sin \varepsilon$, $\quad \triangle D = [5\ 31439]\ A\ B$ [corr. for n_1],

$s' = s$ [corr. for n_1].

$$\text{For} \begin{Bmatrix} \text{partial} \\ \text{total or annular} \end{Bmatrix} \text{phase,} \ \triangle' = \begin{Bmatrix} s' + \sigma \\ s' \sim \sigma \end{Bmatrix};$$

Correction for n_1 to be taken from the table on page 383.

5. $D' = D - \triangle D$, $\quad a' = a - \triangle a$,

$y = (a - \triangle a) \cos D'$, $\quad y_1 = (a_1 - \triangle a_1) \cos D'$,

$x = (D' + a' \text{ corr.}) - \delta$, $\quad x_1 = D_1 - \triangle D_1$.

6. $\tan S = \dfrac{y}{x}$, $\quad \cot \iota = \dfrac{y_1}{x_1}$,

$$W = \frac{y}{\sin S} = \frac{x}{\cos S},$$

$n = W \cos [-(S + \iota)]$, $\quad H = \dfrac{W \cos \iota\ [3.55630]}{y_1}$

7. $\cos \omega = \dfrac{n}{\triangle'}$, $\quad c = \dfrac{H}{\cos \omega}$,

$a = [-(S + \iota)] - \omega$, $\quad b = [-(S + \iota)] + \omega$,

$t_1 = c \sin a$, $\quad t_2 = c \sin b$.

$$\text{Time of} \begin{Bmatrix} \text{beginning} \\ \text{ending} \end{Bmatrix} = T + \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix}.$$

Time of greatest phase $= \frac{1}{2}$ sum of times of beginning and ending.

When $n < s' \sim \sigma$, the eclipse will be total if $s' > \sigma$, or annular if $s' < \sigma$: in this case these last equations No. 7 must be repeated for this phase with $\triangle' = s' \sim \sigma$ the results of which ought to give the same time for the greatest phase.

Take $\triangle'$ for partial phase, and

Portion of sun's disk eclipsed $= \triangle' - n$.

Magnitude of eclipse $= \dfrac{\triangle' - n}{2\sigma}$, the sun's diameter being *unity*.

8. For the positions of the points of contact on the limb of the sun,

$$\text{At} \begin{Bmatrix} \text{beginning} \\ \text{ending} \end{Bmatrix}, \text{angle from north towards east} = \begin{Bmatrix} (-\iota) - \omega \\ (-\iota) + \omega \end{Bmatrix} \text{for } \textit{direct} \text{ image.}$$

$$\text{At} \begin{Bmatrix} \text{beginning} \\ \text{ending} \end{Bmatrix}, \text{angle from north towards east} = \begin{Bmatrix} (180^\circ - \iota) - \omega \\ (180^\circ - \iota) + \omega \end{Bmatrix} \text{for } \textit{inverted} \text{ image.}$$

For the position of the moon's centre at greatest phase,

$$\text{Angle from} \begin{Bmatrix} \text{north} \\ \text{vertex} \end{Bmatrix} \text{towards east} = \begin{Bmatrix} (-\iota) \\ (-\iota) - M \end{Bmatrix} \text{for } \textit{direct} \text{ image.}$$

$$\text{Angle from} \begin{Bmatrix} \text{north} \\ \text{vertex} \end{Bmatrix} \text{towards east} = \begin{Bmatrix} (180^\circ - \iota) \\ (180^\circ - \iota) - M \end{Bmatrix} \text{for } \textit{inverted} \text{ image.}$$

9. For a more accurate calculation of the time, &c., of beginning of the partial phase, assume a convenient time near to the preceding determination. For this time, take out the quantities D, D_1, δ, a, a_1, from the Ephemeris; and proceed as in Nos. 3, 4, 5, 6, 7, omitting b, t_2, and the times of greatest phase and ending.

Let M_1, ι_1, ω_1, be the values of the angles in this computation; then, for the position of the point of contact on the limb of the sun,

$$\text{Angle from} \left\{\begin{matrix}\text{north}\\ \text{vertex}\end{matrix}\right\} \text{towards the east} = \left\{\begin{matrix}(-\iota_1)-\omega_1\\ (-\iota_1)-\omega_1-M_1\end{matrix}\right\} \text{for } \textit{direct} \text{ image.}$$

$$\text{Angle from} \left\{\begin{matrix}\text{north}\\ \text{vertex}\end{matrix}\right\} \text{towards the east} = \left\{\begin{matrix}(180^\circ-\iota_1)-\omega_1\\ (180^\circ-\iota_1)-\omega_1-M_1\end{matrix}\right\} \text{for } \textit{inverted} \text{ image.}$$

10. For a more accurate calculation of the time, &c., of ending of the partial phase, assume a convenient time near to the first determination. For this time, take out the values of D, D_1, δ, a, a_1; and proceed as in Nos. 3, 4, 5, 6, 7, omitting a, t_1, and the times of beginning and greatest phase.

Let M_2, ι_2, ω_2, be the angles in this computation; then, for the position of the point of contact on the limb of the sun,

$$\text{Angle from} \left\{\begin{matrix}\text{north}\\ \text{vertex}\end{matrix}\right\} \text{towards the east} = \left\{\begin{matrix}(-\iota_2)+\omega_2\\ (-\iota_2)+\omega_2-M_2\end{matrix}\right\} \text{for } \textit{direct} \text{ image.}$$

$$\text{Angle from} \left\{\begin{matrix}\text{north}\\ \text{vertex}\end{matrix}\right\} \text{towards the east} = \left\{\begin{matrix}(180^\circ-\iota_2)+\omega_2\\ (180^\circ-\iota_2)+\omega_2-M_2\end{matrix}\right\} \text{for } \textit{inverted} \text{ image.}$$

II.—FORMULÆ FOR REDUCTION TO DIFFERENT PLACES.

11 Instead of Nos. 5, 6, 7, substitute the following:

$$D' = D - \Delta D, \qquad a' = a - \Delta a,$$

$$x_1 = D_1 - \Delta D_1, \qquad y_1 = (a_1 - \Delta a_1) \cos D',$$

$$\tan \iota = \frac{x_1}{y_1}, \qquad k = [3.55630] \frac{\Delta' \cos \iota}{y_1},$$

$$\gamma \cos \psi = \frac{(D + a' \text{ corr.}) - \delta}{\Delta'}, \qquad \gamma \sin \psi = \frac{a \cos D'}{\Delta'},$$

$$p = \gamma \cos (\psi + \iota), \qquad q = k\gamma \sin (\psi + \iota),$$

$$T - q = T'.$$

12. $$b = \frac{[5.31439]\,A}{\Delta'} \text{ [corr. for } n_1],$$

$$e \textit{ in minutes} = [7.9208]\, \Delta a \sin D, \qquad \chi = (90^\circ + \iota) - e.$$

13. H = the true Greenwich hour angle of ☽ at the time T.

$$\frac{L'}{b} = \cos D \cos \iota, \qquad \frac{L''}{kb} = \cos D \sin \iota,$$

$$\frac{\gamma'}{b} \cos (\psi' - H) = \sin D \cos \iota, \qquad \frac{\gamma''}{kb} \cos (\psi'' - H) = \sin D \sin \iota,$$

$$\frac{\gamma'}{b} \sin (\psi' - H) = \cos \chi, \qquad \frac{\gamma''}{kb} \sin (\psi'' - H) = \sin \chi.$$

14. The constants T', k, p, L', L'', γ', γ'', being so computed, the angle ω and the time t of the phase for any place whose north latitude is l and east longitude λ, will be determined by the two following equations, in which the upper sign relates to the beginning and the under sign to the ending.

$$\cos\omega = p - L'\sin l + \gamma'\cos l\cos(\lambda + \psi');$$
$$t = T' \mp k\sin\omega + L''\sin l - \gamma''\cos l\cos(\lambda + \psi'').$$

The result will be the most accurate when the place is near to that on which the previous part of the calculation is founded.

III.—TRANSIT OF MERCURY OR VENUS OVER THE DISK OF THE SUN.

(Same notation for the planet as for the moon.)

15. Assume the time T near to the time of conjunction in longitude, or right ascension.

a = sun's right ascension — planet's right ascension *in arc*;

a_1 = hourly variation of a;

D_1 = sun's hourly motion in declination *minus* that of the planet.

$$\text{For} \begin{Bmatrix}\text{exterior}\\ \text{interior}\end{Bmatrix} \text{contact of limbs, } \Delta = \begin{Bmatrix}\sigma + s,\\ \sigma - s.\end{Bmatrix}$$

For contact of planet's centre with sun's limb, $\Delta = \sigma$:

$$\tan\iota = \frac{D_1}{a_1\cos\delta},$$

$$k = [3.55630]\,\frac{\Delta\cos\iota}{a_1\cos\delta}, \qquad b = \frac{P - \pi}{\Delta},$$

$$\gamma\cos\psi = \frac{(\delta + a\text{ corr.}) - D}{\Delta}, \qquad \gamma\sin\psi = \frac{a\cos\delta}{\Delta},$$

$$\cos \mathrm{w} = \gamma\cos(\psi + \iota), \qquad q = k\gamma\sin(\psi + \iota).$$

16 H = the true Greenwich hour angle of $\odot$ at the time T.

$$k'' = \frac{kb}{\sin \mathrm{w}};$$

$$\frac{L''}{k''} = \cos\delta\cos[(-\iota) \mp \mathrm{w}];$$

$$\frac{\gamma''}{k''}\cos(\psi'' - H) = \sin\delta\cos[(-\iota) \mp \mathrm{w}];$$

$$\frac{\gamma''}{k''}\sin(\psi'' - H) = \sin[(-\iota) \mp \mathrm{w}].$$

17. Then, for the centre of the earth,

$$(t) = (T - q) \mp k\sin \mathrm{w};$$

and, for any place whose latitude is l and east longitude λ,

$$t = (t) \mp [\gamma''\rho\cos l\cos(\lambda + \psi'') - L''\rho\sin l],$$

using the upper signs for the ingress, and the under signs for the egress.

The positions of the points of ingress and egress, estimated from the north point of the sun's limb towards the east, as the transit would be seen from the centre of the earth, will be determined in the same manner as for the immersion and

emersion of an occultation, No. 19, using w for ω. These angles may be assumed to be the same for any place on the surface, the effect of parallax being so very minute.

IV.—OCCULTATION OF A STAR BY THE MOON.

General Limits of Latitude.

18. (a_1 and D_1 at true ☌).

$$\tan \iota = \frac{D_1}{a_1 \cos \delta}, \qquad n = (\text{diff. dec.}) \cos \iota,$$

$$\cos w_1 = \mp \frac{n}{P} - .2725, \qquad \cos w_2 = \mp \frac{n}{P} + .2725,$$

$$\sin \theta = \cos \delta \cos \iota,$$

$$l_1 = w_1 - \theta, \qquad \sin l_2 = \mp \cos \delta \cos (w_2 - \iota),$$

$$w_1,\ w_2,\ \iota,\ \theta, \text{ same sign as } \delta,$$

$$\left.\begin{matrix}\text{upper}\\ \text{under}\end{matrix}\right\} \text{signs when } \delta \text{ is } \left\{\begin{matrix}\text{positive,}\\ \text{negative.}\end{matrix}\right.$$

When w_1 is impossible, $l_1 = 90°$, with the same name as δ.
When w_2 is impossible, l_2 = complement of δ, with different name from δ.

Calculation for particular Place.

19. For the latitude of the place prepare the constants

$$\phi^{(1)} = \rho \cos l, \qquad \phi^{(2)} = \rho \sin l = \frac{\phi^{(1)}}{\cot l}, \qquad \phi^{(3)} = [9.41916]\, \phi^{(1)},$$

which will serve for all occultations at that place.

For the time of true ☌ find

$$h = \text{sidereal time at place} - \text{right ascension of star};$$

and thence determine the time T, as in No. 1. For this time take out the quantities P, s, D, D_1, a, a_1; and compute

$$x = (D - \delta) - (\phi^{(2)} . P \cos \delta - \phi^{(1)} . P \sin \delta \cos h);$$
$$y = a \cos \delta - \phi^{(1)} P \sin h;$$
$$x_1 = D_1 - \phi^{(3)} . P \sin \delta \sin h;$$
$$y_1 = a_1 \cos \delta - \phi^{(3)} . P \cos h.$$

With these proceed as in Nos. 6 and 7, using $\triangle' = s = [9.43537]\, P$.

20. For the positions of the points of immersion and emersion on the limb of the moon,

At $\left\{\begin{matrix}\text{immersion}\\ \text{emersion}\end{matrix}\right\}$, angle from north towards east $= \left\{\begin{matrix}(180° - \iota) - \omega\\ (180° - \iota) + \omega\end{matrix}\right\}$ for *direct* image.

At $\left\{\begin{matrix}\text{immersion}\\ \text{emersion}\end{matrix}\right\}$, angle from north towards east $= \left\{\begin{matrix}(-\iota) - \omega\\ (-\iota) + \omega\end{matrix}\right\}$ for *inverted* image.

For the same angles from the vertex we must deduct the parallactic angle for each time.

21. If an accurate calculation is wanted, proceed as with a solar eclipse.

V.—ECLIPSE OF THE MOON.

22. Fix on a convenient time near to the time of opposition in longitude, or full moon; and for this time find P, s, π, σ,

a = ☽'s right ascension *minus* (☉'s right ascension $\pm$ 12^h), *in arc;*
a_1 = hourly motion of a;
x = (☽'s dec. + a corr.) *plus* ☉'s dec.;
x_1 = hourly motion of x;
y = a cos ☽'s dec.;
y_1 = a_1 cos ☽'s dec.

$$P' = [9.99929]\,P.$$

23.

$$\text{Semid. shadow} = \frac{61}{60}(P' + \pi - \sigma),$$

$$\text{Semid. penumbra} = \frac{61}{60}(P' + \pi - \sigma) + 2\,\sigma.$$

For $\left\{\begin{matrix}\text{external}\\ \text{internal}\end{matrix}\right\}$ contact with shadow, $\Delta' =$ semid. shadow $\left\{\begin{matrix}+\\ -\end{matrix}\right\} s$.

For $\left\{\begin{matrix}\text{external}\\ \text{internal}\end{matrix}\right\}$ contact with penumbra, $\Delta' =$ semid. penumbra $\left\{\begin{matrix}+\\ -\end{matrix}\right\} s$.

The remaining computation as in Nos. 6 and 7.

24. For the positions of the points of contact on the limb of the moon,

At $\left\{\begin{matrix}\text{immersion}\\ \text{emersion}\end{matrix}\right\}$, angle from N. towards E. = $\left\{\begin{matrix}(180° - \iota) - \omega\\ (180° - \iota) + \omega\end{matrix}\right\}$ for *direct* image.

At $\left\{\begin{matrix}\text{immersion}\\ \text{emersion}\end{matrix}\right\}$, angle from N. towards E. = $\left\{\begin{matrix}(-\iota) - \omega\\ (-\iota) + \omega\end{matrix}\right\}$ for *inverted* image.

At the middle of the eclipse,

$\angle$ cent. shadow from N. towards E. = $\left\{\begin{matrix}(180° - \iota)\\ (-\iota)\end{matrix}\right\}$ for $\left\{\begin{matrix}\textit{direct}\\ \textit{inverted}\end{matrix}\right\}$ image.

To get the same angles from the vertex, the parallactic angle must be deducted for the respective times.

Examples.

I.—ECLIPSE OF THE SUN.

Let it be required to calculate the circumstances of the solar eclipse of May 15 1836, as it will be seen at the observatory of Edinburgh.

The elements of this eclipse are stated at page 362.

	h. m. s.			
Greenwich sidereal time at Greenwich mean noon	3 32 58·0			
Longitude	12 43·6 W.			
Edinburgh sidereal time at Greenwich mean noon	3 20 14·4	a_1 .	2′·7	
Sun's right ascension at ☌	3 29 25·2	f .	3·03	
Hour angle h at Greenwich mean noon	— 0 9 10·8		83·1	
Greenwich mean time of ☌ . .	2 21 22·9		·8	
Acceleration	23·2	$a_1 . f$	84·—	
h at ☌	+ 2 13 . .	b +	21	$b^{(1)}$ + 55 (+·87
		$a_1 . f - b$ +	63	50·4
				4·6

	h. m.
Greenwich mean time of true ☌ .	2 21
$b^{(1)} \div (a_1 . f - b)$	+ 52
T	3 13

Constants.

	′ ″		
		ρ .	9·99902
P .	54 23·4	const. .	4·68555
π .	8·5	c .	4·68457
$P - \pi$.	54 14·9	. . . . .	3·51254
log P .	3·51367	A . .	8·19711
const. .	9·43537	cos l .	9·75001
log s .	2·94904	m .	7·94712
σ . .	15′ 49″·9	Q_1 .	4·7172
δ +	18° 58′·5 . .	sin δ +	9·5121
		Q_2 +	2·1764

Computation for $3^h\ 13^m$, Greenwich time.

	° ′ ″		° ′ ″		′ ″
D +	19 33 43	δ +	18 58 29	a +	23 49
D_1 +	9 19			a_1 +	27 43

	h. m. s.
Edinburgh sidereal time at Greenwich mean noon	3 20 14·4
Sidereal equivalent for $3^h\ 0^m$	3 0 29·6
Sidereal equivalent for 13	13 2·1
	6 33 46·1
Moon's right ascension	3 31 9·0
h in time	3 2 37·1
h in arc	+ 45° 39′·3

m	7·94712				
$\cos D$	9·97418	const.	5·31439		
k	7·97294		7·97294		
$\cos h$	+ 9·84446	$\sin h$	+ 9·85440		+ 9·8544
n	+ 7·81740	corr. for n	286		
Q_1	4·7172	log	+ 3·14459	Q_2	+ 2·1764
log	+ 2·5346	Δa	+ 23′ 15″	log	+ 2·0308
Δa_1	+ 5′ 42″			ΔD_1	+ 1′ 47″

h	+ 45° 39′·3				
$\frac{1}{2}\Delta a$	+ 11.6				
(h)	+ 45 50·9	cos	+ 9·84296		+ 9·84296
		$\cot l$	+ 9·83256	$\cos l$	+ 9·75001
θ	+ 25° 20′·9	$\tan \theta$	+ 9·67552	G	+ 9·59297
D	+ 19 33·7	$\sin \theta$	+ 9·63156		
$\theta + D$	+ 44 54·6	cos	+ 9·85017	B	+ 9·81158
			+ 9·78139	check	+ 9·78139
		$\tan (h)$	+ 0·01286		
M	+ 31° 54′·5	$\tan M$	+ 9·79425		
		$\cos M$	+ 9·92885		+ 9·92885
		$\tan (\theta + D)$	+ 9·99864	$\cos \varepsilon$	+ 9·88273
ε	+ 40° 14′·3	$\tan \varepsilon$	+ 9·92749	B	+ 9·81158
		$\cos \varepsilon$	+ 9·88273	const.	+ 5·31439
		$\sin \varepsilon$	+ 9·81022		5·12597
		A	8·19711		8·19711
		n_1	+ 8·00733	corr. for n_1	444
				log	3·32752
		$\log s$	2·94904	ΔD	+ 35′ 26″
			444		
		log	2·95348		
		s'	14′ 58″·4	D_1	+ 9′ 19″
		σ	15 49 ·9	ΔD_1	+ 1 47
		Partial Δ'	30 48 ·3	x_1	+ 7 32
		Annular Δ'	0 51 ·5		

D	+ 19° 33′ 43″	a	+ 23′ 49″	a_1	+ 27′ 43″
ΔD	+ 35 26	Δa	+ 23 15	Δa_1	+ 5 42
D'	+ 18 58 17	a'	+ 0 34		+ 22 1
a' corr.	0	log	+ 1·53148	log	+ 3·12090
δ	+ 18 58 29	$\cos D'$	+ 9·97574		+ 9·97574
x	— 0 12	y	+ 1·50722	y_1	+ 3·09664 (1)

		y . .	$+ 1{\cdot}50722$	y_1 .	$+ 3{\cdot}09664$ (1,
		x . .	$- 1{\cdot}07918$	x_1 .	$+ 2{\cdot}65514$
S . .	$+ 110°\ 28'{\cdot}0$.	tan S .	$- 0{\cdot}42804$	cot ι	$+ 0{\cdot}44150$
		cos S .	$- 9{\cdot}54364$	cos ι	$+ 9{\cdot}97328$
ι . .	$+ 19\ 53{\cdot}5$	W . .	$+ 1{\cdot}53554$	. .	$+ 1{\cdot}53554$
$-(S+\iota)$	$- 130\ 21{\cdot}5$.	cos . .	$- 9{\cdot}81129$	const.	$3{\cdot}55630$
		n . .	$- 1{\cdot}34683$		$+ 5{\cdot}06512$ (2)
	Partial . .	log $\triangle'$.	$3{\cdot}26677$	H	$+ 1{\cdot}96848$ (2) — (1)
ω	$+ 90\ 41{\cdot}1$	cos ω .	$- 8{\cdot}08006$	. .	$- 8{\cdot}08006$
a	$- 221\ \ 2{\cdot}6$	c .	$- 3{\cdot}88842$	c	$- 3{\cdot}88842$
b	$- 39\ 40{\cdot}4$	sin a .	$+ 9{\cdot}81732$	sin b	$- 9{\cdot}80510$
			$- 3{\cdot}70574$		$+ 3{\cdot}69352$

		h. m. s.		h. m. s.	
		$t_1 -$ 1 24 39		$t_2 +$ 1 22 18	
	Assumed time . . .	3 13	. . .	3 13	
	Beginning	1 48 21	Ending	4 35 18	Greenwich mean times
	Longitude	12 44 W.	. . .	12 44 W.	
PARTIAL.	Beginning	1 35 37	Ending	4 22 34	Edinburgh mean times

		n . .	$- 1{\cdot}34683$		
	Annular . .	log $\triangle'$.	$1{\cdot}71181$	H	$+ 1{\cdot}96848$
ω	$+ 115°\ 33'{\cdot}9$	cos ω .	$- 9{\cdot}63502$	. .	$- 9{\cdot}63502$
$-(S+\iota)$	$- 130\ 21\ {\cdot}5$	c .	$- 2{\cdot}33346$	c	$- 2{\cdot}33346$
a	$- 245\ 55\ {\cdot}4$	sin a .	$+ 9{\cdot}96047$	sin b	$- 9{\cdot}40711$
b	$- 14\ 47\ {\cdot}6$		$- 2{\cdot}29393$		$+ 1{\cdot}74057$

		h. m. s.		h. m. s.	
		$t_1 -$ 0 3 17		$t_2 +$ 0 0 55	
	Assumed time . . .	3 13	. . .	3 13	
	Beginning	3 9 43	Ending	3 13 55	Greenwich mean times.
	Longitude	12 44 W.	. . .	12 44 W.	
ANNULAR.	Beginning	2 56 59	Ending	3 1 11	Edinburgh mean times.

POSITIONS OF CONTACTS FOR DIRECT IMAGE.

$$(-\iota)\ -19°{\cdot}9$$
$$\omega\ +90{\cdot}7$$

Partial contact at beginning . . . . . $110°{\cdot}6$ from north towards west.
Partial contact at ending . . . . . . $70{\cdot}8$ from north towards east.

$$(-\iota)\ -19°{\cdot}9$$
$$\omega\ +115{\cdot}6$$

Annular contact at beginning . . . . . $135°{\cdot}5$ from north towards west
Annular contact at ending . . . . . . $95{\cdot}7$ from north towards east.

For the same angles from vertex we must estimate them towards the east, and deduct the angle M, thus

Beginning	$-135°\cdot5$		Ending	$+95°\cdot7$
M	$+31\cdot9$		M	$+31\cdot9$
	$167\cdot4$ towards west.			$63\cdot8$ towards east.

COMPUTATION FOR $1^h\ 48^m$, FOR AN ACCURATE DETERMINATION OF PARTIAL BEGINNING.

D	$+19°\ 19'\ 35''\cdot9$	δ	$+18°\ 57'\ 39''\cdot3$	a	$-15'\ 23''\cdot2$
D_1	$+\ 9\ 26$			a_1	$+27\ 38$

	h.	m.	s.
Edinburgh Sid. Time at Greenwich Mean Noon . .	3	20	14·4
Sidereal Equivalent for $1^h\ 0^m$	1	0	9·9
Sidereal Equivalent for 48		48	7·9
	5	8	32·2
☽'s R. A.	3	28	18·2
h in time	+1	40	14·0
h in arc	+	25°	3′·5

m . .	7·94712				
$\cos D$.	9·97481	const. . .	5·31439		
k . .	7·97231		7·97231		
$\cos h$.	+9·95707	$\sin h$ . .	+9·62690		+9·6269
n . .	7·92938	corr. for n .	370		
Q_1 . .	4·7172	log . . .	+2·91730	Q_2 . . .	2·1764
log . .	+2·6466	Δa . . .	+13′46″·6	log . . .	+1·8033
Δa_1 .	+7′ 23″			ΔD_1 . .	+1′ 4″
h . .	+25° 3′·5				
$\frac{1}{2}\Delta a$.	+6·9				
(h) . .	+25 10·4 .	cos . . .	+9·95666		+9·95666
		$\cot l$ . .	+9·83256	$\cos l$ . .	+9·75001
θ . .	+31° 36′·7	$\tan\theta$ . .	+9·78922	G . . .	+9·70667
D . .	+19 19·6	$\sin\theta$ . .	+9·71946		
$\theta+D$.	+50 56·3 .	cos . . .	+9·79945	B	+9·78665
			+9·92001 .	. check .	+9·92002
		$\tan (h)$ . .	+9.67209		
M_1 . .	+21° 21′·1	$\tan M_1$.	+9·59210		
		$\cos M_1$ . .	+9·96911		+9·96911
		$\tan (\theta+D)$	+0·09068	$\cos \epsilon$ . .	+9·81754
ϵ . . .	+48° 55′·9	tan . . .	+0·05979	B . . .	+9·78665
		cos	+9·81754	const. . .	5·31439
		sin	+9·87733		5·10104
		A	+8·19711		8·19711
		n	+8·07444	corr. for n_1	518
				log . . .	3·30333
				ΔD . .	+33′ 30″·6

log s . .	2·94904		
	518		
log . .	2·95422	D_1 . .	+ 9′ 26″
s′ . . .	15 0·0	ΔD_1 . .	+ 1 4
σ . . .	15 49·9	x_1 . . .	+ 8 22
Δ′ . . .	30 49·9		

	° ′ ″		′ ″		′ ″
D . .	+ 19 19 35·9	α .	− 15 23·2	α_1 .	+ 27 38
Δ D .	+ 33 30·6	Δ α .	+ 13 46·6	Δ α_1 .	+ 7 23
D′ . .	+ 18 46 5·3	α′ .	− 29 9.8		+ 20 15
α′ corr. .	2·2	log .	3·24299	log. .	+ 3·08458
δ . .	+ 18 57 39·3	cos D′ .	+ 9·97627	. . .	+ 9·97627
x . .	− 11 31·8	y . .	− 3·21926	y_1 . .	+ 3·06085 (1)
		x . .	− 2·83998	x_1 . .	+ 2·70070
	° ′	tan S .	+ 0·37928	cot ι_1 .	+ 0·36015
S . .	− 112 39·81				
		sin S .	− 9·96510	cos ι_1 .	+ 9·96215
ι . .	+ 23 34·49	W . .	+ 3·25416	. . .	+ 3·25416
−(S+ι)	+ 89 5·32	cos .	+ 8·20168	const. .	3·55630
					6·77261 (2)
		n . .	+ 1·45584		
		log Δ′.	+ 3·26715	H . .	+ 3·71176 (2) − (1)
ω_1	+ 89 6·93	cos ω_1 .	+ 8·18869	. . .	+ 8·18869
α	− 0 1·61			c . .	+ 5·52307
			sin α	sin 1′	6·46373
				− 1·61	− 0·20683
					− 2·19363
				t_1 . .	− 0h 2m 36s
				Assumed time	1 48 0
				Beginning .	1 45 24 Greenh M.T.
				Long. . .	12 44 W.
		Partial.		Beginning .	1 32 40 Edin. M.T

If the calculation be repeated for the Greenwich time 1h 45m, it will lead to exactly the same result, which is therefore to the accurate second, according to the data employed.

Position of Contact for Direct Image.

	°
$(-\iota_1)$	− 23·6
ω_1	+ 89·1
$(-\iota_1) - \omega_1$	− 112·7
M_1	+ 21·4
$(-\iota_1) - \omega_1 - M_1$ . . .	− 134·1

The point of contact is therefore $\left\{ \begin{matrix} 113° \\ 134 \end{matrix} \right\}$ from $\left\{ \begin{matrix} \text{north} \\ \text{vertex} \end{matrix} \right\}$ towards west.

II.—EQUATIONS FOR REDUCTION OF PARTIAL BEGINNING.

The data for this computation are taken from the preceding one.

	3·55630		5·31439		7·9208		
Δ'	3·26715	A .	8·19711	Δa .	+ 2·9173		
cos ι	9·96215	corr. for n_1	518	sin D	+ 9·5198		° ′
	6·78560		3·51668		+ 0·3579 . . e	+	0 2·3
ν_1	3·06085	Δ'	3·26715		90° + ι .		113 34·5
k	+ 3·72475	b .	+ 0·24953		χ .		113 32·2
		k .	+ 3·72475				
		kb .	+ 3·97428				

	° ′ ″		
D	+ 19 19 35·9	a	− 2·96530
a' corr.	2·2	cos D'	+ 9·97627
δ	+ 18 57 39·3	$\Delta'\gamma$ sin ψ	− 2·94157
	+ 21 58·8	$\Delta'\gamma$ cos ψ	+ 3·12018
ψ	− 33 32·2	tan ψ	− 9·82139
ι	+ 23 34·5	cos ψ	+ 9·92092
$\psi + \iota$	− 9 57·7	$\Delta'\gamma$ . .	+ 3·19926
		Δ' . .	3·26715
γ	9·93211	γ . .	9·93211
cos ($\psi + \iota$)	+ 9·99341	sin ($\psi + \iota$)	− 9·23802
	+ 9·92552	k . .	3·72475
p	+ 0·84240		− 2·89488
h	+ 25° 3′·5	q . .	− 0h 13m 5s
Long.	3 10·9 W.	T . .	+ 1 48
H	+ 28 14·4	T' . .	+ 2 1 5

	cos D	+ 9·97481		cos D	+ 9·97481
	cos ι	+ 9·96215		sin ι	+ 9·60200
	b .	+ 0·24953		kb .	+ 3·97428
	L' .	+ 0·18649		L'' .	+ 3·55109
	sin D	+ 9·51977		sin D	+ 9·51977
	cos ι	+ 9·96215		sin ι	+ 9·60200
		+ 9·48192			+ 9·12177
	cos χ	− 9·60134		sin χ	+ 9·96228
$\psi' - H$ − 52° 46′·8	tan .	− 0·11942	$\psi'' - H$ + 81° 47′·1	tan .	+ 0·84051
	sin .	− 9·90109		sin .	+ 9·99552
H + 28 14·4		+ 9·70025	H + 28 14·4		+ 9·96676
ψ' . − 24 32·4	b .	0·24953	ψ' + 110 1·5	kb .	3·97428
	γ' .	+ 9·94978		γ'' .	+ 3·94104

We have hence, for the Greenwich time t of beginning, at any place whose latitude is l, $=$ north, $-$ south, and longitude λ, $+$ east, $-$ west, the two following equations, which may be safely depended on for any place in Scotland or the North of England.

$$\cos\omega = 0\cdot 84240 - [0\cdot 18649]\sin l + [9\cdot 94978]\cos l\cos(\lambda - 24^\circ\, 32'\cdot 4)$$
$$t = 2^h\, 1^m\, 5^s - [3\cdot 72475]\sin\omega + [3\cdot 55109]\sin l - [3\cdot 94104]\cos l\cos(\lambda + 110^\circ\, 1'\cdot 5)$$

Contact on ☉'s limb, $\omega + 23^\circ\, 34'\cdot 5$ from the north towards the west.

As a check on this calculation take the assumed radical place, Edinburgh, and $l = +55^\circ\, 46'\cdot 9$, $\lambda = -3^\circ\, 10'\cdot 9$, giving $\omega = 89^\circ\, 6'\cdot 9$ and $t = 1^h\, 45^m\, 24^s$, which perfectly coincide with the results of the original calculation.

Similar calculations for the ending of the eclipse give the equations,

$$\cos\omega = 0\cdot 93848 - [0\cdot 20291]\sin l + [9\cdot 88677]\cos l\cos(\lambda + 27^\circ\, 6'\cdot 7)$$
$$t = 1^h\, 38^m\, 33^s + [3\cdot 66890]\sin\omega + [3\cdot 35544]\sin l - [3\cdot 90073]\cos l\cos(\lambda + 153^\circ\, 3'\cdot 8)$$

Contact on ☉'s limb, $\omega - 16^\circ\, 56'\cdot 2$ from the north towards the east.

Also by calculating with $T = 3^h\, 13^m$ for the annular phase there will result

$$\cos\omega = 29\cdot 66600 - [1\cdot 75159]\sin l + [1\cdot 46950]\cos l\cos(\lambda + 1^\circ\, 42'\cdot 4)$$
$$t = 1^h\, 43^m\, 7^s \mp [2\cdot 14475]\sin\omega + [3\cdot 45484]\sin l - [3\cdot 92550]\cos l\cos(\lambda + 131^\circ\, 55'\cdot 9)$$

Contact on ☉'s limb, $-19^\circ\, 53'\cdot 5 \mp \omega$ from the north towards the east, the upper sign appertaining to the beginning and the under sign to the ending. If $\cos\omega > 1$, the place will be without the limits, and the eclipse will not be annular.

By taking $l = +55^\circ\, 46'\cdot 9$, $\lambda = -3^\circ\, 10'\cdot 9$, the results will exactly correspond with the special calculation.

Note.—The expression of $\cos\omega$ for the annular phase, as the appearance of this phase is comprised within narrow limits on the surface of the earth, will afford a very convenient and simple determination of the places which range in those limits as well as those which range in the central line; and we may expect very accurate results throughout the portion of country originally taken into consideration. Thus for the southern limit we must obviously have $\cos\omega = +1$, for the central line $\cos\omega = 0$, and for the northern limit $\cos\omega = -1$; and hence the following conditions:

$$p - L'\sin l + \gamma'\cos l\cos(\lambda + \psi') = \begin{cases} +1 \\ 0 \\ -1 \end{cases} \text{for} \begin{cases} \text{southern limit.} \\ \text{central eclipse.} \\ \text{northern limit.} \end{cases}$$

By making the assumptions

$$\left.\begin{aligned} n'\cos N' &= \gamma'\cos(\lambda + \psi') \\ n'\sin N' &= L' \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (r)$$

they will give

$$n'\cos(N' + l) = \begin{cases} -p + 1 \\ -p \\ -p - 1 \end{cases} \text{for} \begin{cases} \text{southern limit} \\ \text{central eclipse} \\ \text{northern limit} \end{cases} \quad \ldots\ldots \quad (s)$$

If we therefore take any meridian whose east longitude is λ, these two equations (r), (s) will serve to determine the extreme latitudes l, on this meridian, between which the eclipse will be annular as well as that where it will be central.

For the preceding eclipse, these equations will be

$$n'\cos N' = [1\cdot 46950]\cos(\lambda + 1^\circ\, 42'\cdot 4),$$
$$n'\sin N' = [1\cdot 75159];$$

$$n'\cos(N' + l) = \begin{cases} -[1\cdot 45737] \\ -[1\cdot 47226] \\ -[1\cdot 48665] \end{cases} \text{for} \begin{cases} \text{southern limit.} \\ \text{central eclipse.} \\ \text{northern limit.} \end{cases}$$

If we take, for example, the meridian of Edinburgh, and use $\lambda = -3^\circ\ 10'\cdot 9$, there will result,

	° ′
Extreme southern point of annular appearance, N.	54 19·7
Point of central appearance, N.	55 20·4
Extreme northern point of annular appearance, N.	56 21·7

which are geocentric latitudes.

III. CALCULATION OF THE TRANSIT OF MERCURY, November 7, 1835.

The conjunction in right ascension takes place about $7^h\ 38^m$; take therefore $T = 7^h\ 40^m$, and we readily find from the ephemeris the following data:

$$\delta - 16^\circ\ 15'\ 58''\cdot 2$$

D	$-16^\circ\ 22'\ 4''\cdot 2$	a	$+\ 0'\ 10''\cdot 95$
D_1	$-\ 2'\ 32''\cdot 6$	a_1	$+\ 5'\ 32''\cdot 7$
s	$4''\cdot 8$	σ	$16'\ 10''\cdot 4$
P	$12''\cdot 66$	π	$8''\cdot 66$

With these quantities, the calculation, for external contact of limbs, is as follows:

$$\begin{array}{llll}
\sigma & 16'\ 10''\cdot 4 & P & 12''\cdot 66 \\
s & 4\cdot 8 & \pi & 8\cdot 66 \\
\Delta & 16\ 15\cdot 2 & & 4\cdot 00\ \ldots\ 0\cdot 60200 \\
 & & \Delta & 2\cdot 98909 \\
 & & b & +\ 7\cdot 61297
\end{array}$$

$$\begin{array}{llll}
a & +\ 1\cdot 03941 & a_1 & +\ 2\cdot 52205 \\
\cos\delta & +\ 9\cdot 98226\ \ldots & & +\ 9\cdot 98226 \\
a\cos\delta & +\ 1\cdot 02167 & a_1\cos\delta & +\ 2\cdot 50431
\end{array}$$

$$\begin{array}{ll}
\delta & -16^\circ\ 15'\ 58''\cdot 2 \\
s\ \text{corr.} & \cdot 0 \\
D & -16\ 22\ 4\cdot 2 \\
 & +\ 6\ 6\cdot 0 \\
\psi & +\ 1^\circ\ 38'\cdot 7 \\
\iota & -25\ 32\cdot 3 \\
\psi+\iota & -23\ 53\cdot 6
\end{array}
\qquad
\begin{array}{ll}
a\cos\delta & +\ 1\cdot 02167 \\
 & +\ 2\cdot 56348 \\
\tan\psi & +\ 8\cdot 45819 \\
\cos\psi & +\ 9\cdot 99982 \\
 & +\ 2\cdot 56366 \\
\Delta & 2\cdot 98909 \\
\gamma & +\ 9\cdot 57457
\end{array}
\qquad
\begin{array}{ll}
D_1 & -\ 2'\ 32''\cdot 6\ \ldots\ -2\cdot 18355 \\
a_1\cos\delta & +\ 2\cdot 50431 \\
\iota\ -25^\circ\ 32'\cdot 3\ \ \tan & -\ 9\cdot 67924 \\
\cos & +\ 9\cdot 95535 \\
\Delta & 2\cdot 98909 \\
\text{const.} & 3\cdot 55630 \\
 & +\ 6\cdot 50074 \\
k & +\ 3\cdot 99643 \\
b & +\ 7\cdot 61297 \\
k\,b & +\ 1\cdot 60940 \\
\sin w & +\ 9\cdot 97278 \\
k'' & +\ 1\cdot 63662 \\
\cos\delta & +\ 9\cdot 98226 \\
k''\cos\delta & +\ 1\cdot 61888
\end{array}$$

$$\begin{array}{ll}
\gamma & +\ 9\cdot 57457 \\
\cos(\psi+\iota) & +\ 9\cdot 96109 \\
\cos w & +\ 9\cdot 53566 \\
w & +\ 69^\circ\ 55'\cdot 4 \\
 & \\
\sin w & +\ 9\cdot 97278 \\
k & +\ 3\cdot 99643 \\
k\sin w & +\ 3\cdot 96921\ \ldots
\end{array}
\qquad
\begin{array}{ll}
k & +\ 3\cdot 99643 \\
\sin(\psi+\iota) & -\ 9\cdot 60749 \\
 & -\ 3\cdot 17849 \\
 & \text{h. m. s.} \\
q & -\ 0\ 25\ 8\cdot 3 \\
T & +\ 7\ 40 \\
T-q & +\ 8\ 5\ 8\cdot 3 \\
 & \\
 & \\
 & 2\ 35\ 15.6
\end{array}$$

Mean time of $\left\{\begin{array}{ll}\text{ingress} & 5\ 29\ 52\cdot 7 \\ \text{egress} & 10\ 40\ 23\cdot 9\end{array}\right\}$ for the centre of the earth.

Constants for Reduction of Ingress.

		h. m. s.		
		5 29 52·7		
	Equa. +	16 10·0		
H in	time +	5 46 2·7		
	arc +	86° 30′·7		
	$-\iota$ +	25 32·3		
	w	69 55·4		
	$-\iota$ − w −	44 23·1	cos + 9·85410	sin − 9·84477
			sin δ − 9·44733	
			− 9·30143	− 9·30143
	$\psi'' - H$ −	105 58·3		tan + 0·54334
				sin − 9·98290
	ψ'' −	19 27·6		
				+ 9·86187
			$k'' \cos\delta$ + 1·61888	k'' + 1·63662
			L'' + 1·47298	γ'' + 1·49849

Constants for Reduction of Egress.

		h. m. s.		
		10 40 23·9		
	Equa. +	16 9·2		
H in	time +	10 56 33·1		
	arc +	164° 8′·3		
	$-\iota$ + w +	95 27·7	cos − 8·97854	sin + 9·99802
			sin δ − 9·44733	
			+ 8·42587	+ 8·42587
	$\psi'' - H$ +	88 28·0		tan + 1·57215
				sin + 9·99984
ψ''	+	252 36·3		
	−	107 23·7		+ 9·99818
			$k'' \cos\delta$ + 1·61888	k'' + 1·63662
			L'' − 0·59742	γ'' + 1·63480

The former part of the calculation repeated for the times $5^h\ 30^m$ and $10^h\ 40^m$ we shall find more accurate times of ingress and egress, for the centre of the earth, to be $5^h\ 29^m\ 56^s$ and $10^h\ 40^m\ 31^s$, which, however, still cannot be depended on within a few seconds. More reliance can be placed in the amount of reduction for parallax. The times reduced for any place whose north latitude is l, and east longitude λ, viz.:

Ingress, Nov. $7^d\ 5^h\ 29^m\ 56^s + [1{\cdot}4730]\ \rho \sin l - [1{\cdot}4985]\ \rho \cos l \cos(\lambda - 19°\ 28')$
Egress, " " $10\ 40\ 31 + [0{\cdot}5974]\ \rho \sin l + [1{\cdot}6348]\ \rho \cos l \cos(\lambda - 107°\ 24')$

will indicate, with considerable accuracy, the difference between the times at any two places.

The positions of the contacts on the sun's limb, for an *inverted* image, will be

Contact at { ingress . . . . . . 44° 23′ / egress . . . . . . 95 28 } from the north towards the { west. / east.

IV.—OCCULTATION OF A STAR.

On January 7, 1836, the star ι Leonis, whose right ascension is $10^h\ 23^m\ 26^s \cdot 4$ and declination N. $14°\ 58'\ 39''$, will be occulted by the moon.

Limits of Latitude.

At the time of true ☌ in right ascension, viz., $12^h\ 12^m\ 17^s$, we have the following data:

	° ′ ″		′ ″
$D +$	15 33 2	$D_1 -$	11 47
$\delta +$	14 58 39	$a_1 +$	30 41
$D - \delta +$	0 34 23	$P +$	56 4

with which we proceed thus:

$D_1 - 11'\ 47''$	. $-2 \cdot 84942$			$-\frac{n}{P} -\ \cdot 5699$
$a_1 + 30\ 41$	. $+3 \cdot 26505$			const. $+\ \cdot 2725$
	$-9 \cdot 58437$		° ′	
$\delta + 14° 59'$	cos $+ 9 \cdot 98498$	$w_1 +$	147 24	. nat. cos $-\ \cdot 8424$
$\iota - 21\ 41$	tan $-9 \cdot 59939$	$w_2 +$	107 18	. nat. cos $-\ \cdot 2974$
	cos $+ 9 \cdot 96813$	$\iota +$	21 41	. log. cos $+ 9 \cdot 9681$ (1)
diff. dec. $+ 34'\ 23''$	. $+ 3 \cdot 31450$	$+$	86 37	. log. cos $+ 8 \cdot 8833$ (2)
				log. cos $\delta + 9 \cdot 9850$ (3)
	$n + 3 \cdot 28263$			
$P + 56'\ 4''$	. $+ 3 \cdot 52686$	$\theta +$	63 51	. log. cos $+ 9 \cdot 9531$ (1)+(3)
$\frac{n}{P} + \cdot 5699$	. $+ 9 \cdot 75577$	$l_1 +$	83 33	
		$l_2 -$	4 14	— log. sin $l_2 + 8 \cdot 8683$ (2)+(3)

The star may therefore be occulted between the parallels of latitude N. 83° 33′ and S. 4° 14′. The parallel of Greenwich is within these limits; and if the hour angle of the star be computed roughly for the meridian of Greenwich, the star will be found to be considerably elevated above the horizon. A special calculation for the observatory of Greenwich will consequently serve as an example of the circumstances for a particular place.

Calculation for Greenwich Observatory.

Constants $\phi^{(1)}$, $\phi^{(2)}$, $\phi^{(3)}$.

ρ	$9 \cdot 99913$		
cos l . . .	$+ 9 \cdot 79610$		
$\phi^{(1)}$ . . .	$+ 9 \cdot 79523$	. . . .	$+ 9 \cdot 79523$
cot l . . .	$+ 9 \cdot 90381$	const. . .	$9 \cdot 41916$
$\phi^{(2)}$ . . .	$+ 9 \cdot 89142$	$\phi^{(3)}$ . .	$+ 9 \cdot 21439$

These will be constant for all occultations at Greenwich.

	h. m. s.		h. m. s.
Sidereal time at mean noon .	19 4 22·4		
Star's right ascension . .	10 23 26·4		
h at mean noon	— 15 19 4.0	. . . .	— 15 19 4·0
Mean time of true ☌ . .	+ 12 12	T	+ 11 6
Acceleration	+ 2	acceleration .	+ 1 49·4
h at true ☌	— 3 5	h in time . .	— 4 11 14·6
		h in arc . .	— 62° 48′·7

With this and $a_1 = 30' \cdot 7$ we find, by the table at p. 405, $T = 11^h\ 6^m$.

h at mean noon is put down negatively, in order to have more readily the other values of h less than 12^h or $180°$.

P 56′ 4″	+ 3·52686		+ 3.52686		+ 3·52686
$\phi^{(2)}$	+ 9·89142	cos h	+ 9·65983	sin h	− 9·94915
	+ 3·41828		+ 3·18669		− 3·47601
cos δ	+ 9·98499	sin δ	+ 9·41236	sin δ	+ 9·41236
	+ 3·40327		+ 2·59905		− 2·88837
	+ 42′ 11″	$\phi^{(1)}$	+ 9·79523	$\phi^{(3)}$	+ 9·21439
	+ 4 8		+ 2·39428		− 2·10276
	+ 38 3				− 2′ 7″
$D-\delta$	+ 47 22			D_1	− 11 42
ι	+ 9 19			x_1	− 9 35
a	− 33′ 54″			a_1	+ 30′ 44″
	− 3·30835				= 3·26576
cos δ	+ 9·98499			cos δ	+ 9·98499
	− 3·29334				+ 3·25075
	− 32′ 45″				+ 29′ 41″
P sin h	− 3·47601	P	3·52686	P cos h	+ 3·18669
$\phi^{(1)}$	+ 9·79523	const.	9·43537	$\phi^{(3)}$	+ 9·21439
	− 3·27124	Δ'	2·96223		+ 2·40108
	− 31′ 7″				+ 4′ 12″
y	− 1′ 38″			y_1	+ 25 29
	− 1·99123				+ 3·18441
x	+ 2·74741			x_1	− 2·75967
tan S	− 9·24382	S	− 9° 56′·6	cot ι	− 0·42474
cos S	+ 9·99343	ι	− 20 36·6	cos ι	+ 9·97128
W	+ 2·75398			W	+ 2·75398
cos $-(S+\iota)$	+ 9·93508	$-(S+\iota)$	+ 30 33·2		3·55630
n	+ 2·68906				+ 6·28156
Δ'	2·96223			H	+ 3·09715
cos ω	+ 9·72683	ω	+ 57 47·0	cos ω	+ 9·72683
c	+ 3·37032	a	− 27 13·8	c	+ 3·37032
sin a	− 9·66045	b	+ 88 20·2	sin b	+ 9·99982
	− 3·03077				+ 3·37014

t_1	− 0h 17m.9	t_2	+ 0h 39m.1
T	11 6	T	11 6
Immersion	10 48 .1	Emersion	11 45 1 mean times.
Acceleration	1 ·8	Acceleration	2 ·0
S. T. mean noon	19 4 ·4	S. T. mean noon	19 4 ·4
Immersion	5 54 ·3	Emersion	6 51 ·5 . sid. times.
Star's R. A.	10 23 ·4	Star's R. A.	10 23 ·4
Im. h	− 4 29·1 = − 67°	Em. h =	− 3 31 ·9 = − 53°
Parallactic ∠	− 39°·7	Parallactic ∠	− 36°·9
$(-\iota)$	+ 20 ·6	$(-\iota)$	+ 20 ·6
ω	+ 57 8	ω	+ 57 ·8
From north	− 37 2 to the east.	From north	+ 78 ·4 to the east.
vertex	+ 2 5	vertex	+ 115 ·3

These angles are for the *inverted* image; and, being estimated towards the east, the negative values must be considered as towards the west. The declination of the star gives for the latitude of Greenwich a semi-diurnal arc of $7^h\ 23^m$; as this exceeds the value of h both at immersion and emersion, the immersion and emersion will both occur above the horizon.

V.—CALCULATION OF THE ECLIPSE OF THE MOON, April 30, 1836.

The opposition or full moon takes place at $19^h\ 58^m$. For the computation assume the time $20^h\ 0^m$.

	19^h	20^h	21^h
	h. m. s.	h. m. s.	h. m. s.
☽'s R. A.	14 32 51·35	14 35 11·19	14 37 31·43
☉'s R. A. + 12^h	14 33 52·38	14 34 1·91	14 34 11·45
a in time	− 1 1·03	+ 1 9·28	+ 3 19·98
a in space	− 15′ 15″	+ 17′ 19″	+ 50′ 0″

$a = \begin{matrix} -15' & 15'' \\ +17 & 19 \\ +50 & 0 \end{matrix} \quad \begin{matrix} +32' & 34'' \\ +32 & 41 \end{matrix} \qquad a_1 = +32'\ 38''$

	19^h	20^h	21^h
	° ′ ″	° ′ ″	° ′ ″
☽'s dec.	− 14 5 19	− 14 19 58	− 14 34 32
a cor.	0	1	5
☉'s dec.	+ 15 6 35	+ 15 7 20	+ 15 8 6
x	+ 1 1 16	+ 47 21	+ 33 29

$x = \begin{matrix} +61' & 16'' \\ +47 & 21 \\ +33 & 29 \end{matrix} \quad \begin{matrix} -13' & 55'' \\ -13 & 52 \end{matrix} \qquad x_1 = -13'\ 54''$

		a + 3·01662	a_1 + 3·29181	$P = 60'\ 19''$
		cos D + 9·98627	. . + 9·98627	
		y + 3·00289	y_1 + 3·27808	P 3·55859
		x + 3·45347	x_1 − 2·92117	9·99929
				3·55788
S	+ 19° 30′·7	tan S + 9·54942	cot ι − 0·35691	
		cos S + 9·97431	cos ι + 9·96163	P' . 60′ 13″
				π . 9
ι	− 23 43 ·9	W + 3·47916	. . + 3·47916	σ . 15 53
−(S+ι)	+ 4 13 ·2	cos + 9·99882	3·55630	44 29
		n + 3·47798	+ 6·99709	$\frac{1}{60}$. 44
		External . . △′ 3·56808	H + 3·71901	45 13 SHADOW
				s 16 26
ω	+ 35 38 ·7	cos ω + 9·90990	. . + 9·90990	
a	− 31 25 ·5	c + 3·80911	c + 3·80911	△′ 61 39 external
b	+ 39 51 ·9	sin a − 9·71716	sin b + 9·80685	△′ 28 47 internal
		− 3·52627	+ 3·61596	
		t_1 − $56^m.0$	t_2 + $1^h\ 8^m.8$	
		Assumed time 20^h 0	. . . 20 0	
		Beginning 19 4 ·0	Ending 21 8 ·8 Greenw^h mean times.	

For the times at any other place, it will only be necessary to take into account the difference of longitude.

The positions of the points of contact on the limb of the moon may be determined in the same manner as those of an occultation, and will here be unnecessary.

As Δ' for internal contact with shadow is less than n, no internal contact can take place, and therefore the eclipse is only partial.

The contacts with the penumbra are to be determined in a similar manner from the same values of n, H, and will also be unnecessary here.

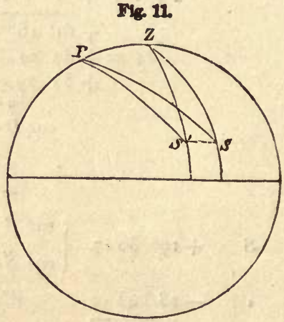
Δ' for external contact with shadow	61′	39″
n	50	6
Eclipsed	11	33

which divided by $2s = 32' 52''$, gives 0·351 for the magnitude of the eclipse, the moon's diameter being *unity.*

APPENDIX XII.

EQUATION OF EQUAL ALTITUDES.

Fig. 11.



Let P be the pole, Z the zenith, S' the place of the sun in the afternoon, S the place he would have occupied had his declination or polar distance PS remained unchanged. Make

$l =$ latitude of place $= 90° - PZ$;
$x =$ declination of sun $= 90° - PS$;
$a =$ altitude of sun $= 90° - ZS$;
$P =$ hour angle ZPS;

then in the triangle ZPS,

$$\sin a = \sin l \sin x + \cos l \cos x . \cos P \quad . \quad . \quad . \quad (a)$$

Differentiating, supposing x and P alone to vary, we have

$$dx . \cos x . \sin l = \cos l . \cos x . \sin P . dP + \cos l . \cos P . \sin x \, dx,$$

or

$$dx . (\cos x . \sin l - \cos l . \cos P . \sin x) = dP . \sin P \cos l \cos x ;$$

whence

$$dP = dx . \left(\frac{\tan l}{\sin P} - \frac{\tan x}{\tan P}\right) . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Denote by δ the change in declination from the next preceding to the next following noon or change in 48 hours, and by t the interval in hours between the epochs of equal altitudes in the morning and afternoon. Then

$$48 : \delta :: t : d\,x,$$

whence

$$d\,x = \frac{\delta\,.\,t}{48};$$

also

$$\tfrac{1}{15}.P = \tfrac{1}{2}\,t,$$

which substituted in Eq. (b) give

$$d\,P = \frac{\delta\,.\tan l\,.\,t}{48\,.\sin 7\tfrac{1}{2}\,t} - \frac{\delta\,.\tan x\,.\,t}{48\,.\tan 7\tfrac{1}{2}\,t};$$

converting both members into time and taking one half, we have, after writing d for x, and making $t_{,} = \frac{1}{2} \times \frac{1}{15}\,.\,d\,P = \frac{1}{30}\,d\,P$,

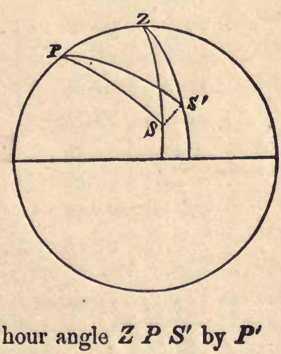
$$t_{,} = \frac{\delta\,.\tan l\,.\,t}{1440\,.\sin 7\tfrac{1}{2}\,t} - \frac{\delta\,.\tan d\,.\,t}{1440\,.\tan 7\tfrac{1}{2}\,t}\,. \quad . \quad . \quad . \quad . \qquad (c)$$

as in the text, page 187.

APPENDIX XIII.

CORRECTION FOR DIFFERENCES OF REFRACTION.

Fig. 12.



Let P be the pole, Z the zenith, and S the place of the sun had the air undergone no change, and S' the place as determined by a change of atmospheric refraction. Then, employing the same notation as in the preceding appendix and resuming its equation (a), regarding the altitude a as referring to the place S and P to the hour angle $Z\,P\,S$, we have, writing d for x,

$$\sin a = \sin l\,.\sin d + \cos l\,.\cos d\,.\cos P,$$

and denoting the altitude of S' by a' and the hour angle $Z\,P\,S'$ by P'

$$\sin a' = \sin l\,.\sin d + \cos l\,.\cos d\,.\cos P';$$

and by subtraction,

$$\sin a' - \sin a = \cos l\,.\cos d\,(\cos P' - \cos P);$$

but $$\sin a' - \sin a = 2 \sin \tfrac{1}{2}(a' - a) \,.\, \cos \tfrac{1}{2}(a' + a),$$
$$\cos P' - \cos P = 2 \sin \tfrac{1}{2}(P - P') \,.\, \sin \tfrac{1}{2}(P' + P);$$

whence by substitution,

$$\sin \tfrac{1}{2}(a' - a) \,.\, \cos \tfrac{1}{2}(a' + a) = \cos l \,.\, \cos d \,.\, \sin \tfrac{1}{2}(P - P') \,.\, \sin \tfrac{1}{2}(P' + P),$$

and because a and a' as also P' and P differ by very small quantities, the above becomes, by transposing and dividing,

$$P - P' = \frac{(a' - a) \,.\, \cos a}{\cos l \,.\, \cos d \,.\, \sin P}.$$

But denoting the refraction in the afternoon by r' and that in the morning by r, we have

$$a' - a = r' - r;$$

substituting and converting both members into time, and writing $t_{\prime\prime}$ for the first member, we have

$$t_{\prime\prime} = \tfrac{1}{15} \,.\, \frac{(r' - r) \,.\, \cos a}{\cos l \,.\, \cos d \,.\, \sin P},$$

as in the text at page 188.

TABLES.

TABLE I.

Mr. Ivory's Mean Refractions; with the Logarithms and their Differences annexed.

Zenith Dist.	Mean Refraction.	Log.	Diff.	Zenith Dist.	Mean Refraction.	Log.	Diff.
°	′ ″			°	′ ″		
1	0 1·02	0·0085	3012	25	0 27·24	1·4352	201
2	2·04	0·3097	1763	26	28·49	1·4547	195
3	3·06	0·4860	1252	27	29·76	1·4736	189
4	4·08	0·6112	974	28	31·05	1·4921	185
5	5·11	0·7086	796	29	32·38	1·5102	181
6	6·14	0·7882	675	30	33·72	1·5279	177
7	7·17	0·8557	587	31	35·09	1·5452	173
8	8·21	0·9144	519	32	36·49	1·5622	170
9	9·25	0.9663	466	33	37·93	1·5790	168
10	10·30	1·0129	424	34	39·39	1·5954	164
11	11·35	1·0553	388	35	40·89	1·6116	162
12	12·42	1·0941	359	36	42·42	1·6276	160
13	13·49	1·1300	334	37	44·00	1·6435	159
14	14·56	1·1634	313	38	45·61	1·6591	156
15	15·66	1·1947	294	39	47·27	1·6746	155
16	16·75	1·2241	278	40	48·99	1·6901	155
17	17·86	1·2519	265	41	50·75	1·7055	154
18	18·98	1·2784	252	42	52·57	1·7207	152
19	20·11	1·3036	241	43	54·43	1·7358	151
20	21·26	1·3277	230	44	56·35	1·7510	152
21	22·42	1·3507	222	45	58·36	1·76611	151
22	23·60	1·3729	215	46	1 0·43	1·78123	1512
23	24·80	1·3944	207	47	2·57	1·79637	1514
24	0 26·01	1·4151		48	1 4·80	1·81155	1518

TABLE I.—(Continued.)

Zenith Dist.	Mean Refraction.	Log.	Diff.	Zenith Dist.	Mean Refraction.	Log.	Diff.
° ′	′ ″			° ′	′ ″		
49 0	1 7·11	1·82678	1523	72 30	3 3·23	2·26299	429
50 0	9·52	1·84208	1530	40	5·06	2·26732	433
51 0	12·02	1·85747	1539	50	6·93	2·27168	436
52 0	14·64	1·87298	1551	73 00	8·83	2·27608	440
53 0	17·38	1·88863	1565	10	10·77	2·28051	443
54 0	20·24	1·90440	1577	20	12·74	2·28498	447
55 0	23·25	1·92036	1596	30	14·75	2·28948	450
56 0	26·41	1·93653	1617	40	16·80	2·29402	454
57 0	29·73	1·95291	1638	50	18·88	2·29860	458
58 0	33·23	1·96955	1664	74 00	21·01	2·30322	462
59 0	36·93	1·98646	1691	10	23·18	2·30789	467
60 0	40·85	2·00368	1722	20	25·39	2·31259	470
61 0	45·01	2·02124	1756	30	27·66	2·31734	475
62 0	49·44	2·03918	1794	40	29·95	2·32213	479
63 0	54·17	2·05754	1836	50	32·30	2·32696	483
64 0	59·22	2·07635	1881	75 00	34·70	2·33184	488
65 0	2 4·65	2·09567	1932	10	37·16	2·33677	493
66 0	10·48	2·11555	1988	20	39·65	2·34174	497
67 0	16·78	2·13603	2048	30	42·21	2·34676	502
68 0	23·61	2·15719	2116	40	44·82	2·35183	507
69 0	31·04	2·17910	2191	50	47·48	2·35695	512
70 00	39·16	2·20185	2275	76 00	50·21	2·36212	517
10	40·59	2·20573	388	10	53·00	2·36735	523
20	42·04	2·20963	390	20	55·85	2·37263	528
30	43·52	2·21356	393	30	58·76	2·37796	533
40	45·02	2·21752	396	40	4 1·74	2·38334	538
50	46·53	2·22150	398	50	4·79	2·38879	545
71 00	48·08	2·22552	402	77 00	7·91	2·39430	551
10	49·65	2·22956	404	10	11·11	2·39987	557
20	51·25	2·23363	407	20	14·39	2·40550	563
30	52·87	2·23773	410	30	17·74	2·41119	569
40	54·53	2·24186	413	40	21·19	2·41695	576
50	56·21	2·24603	417	50	24·72	2·42278	583
72 00	57·92	2·25022	419	78 00	28·33	2·42867	589
10	59·66	2·25445	423	10	32·04	2·43463	596
20	3 1·43	2·25870	425	20	4 35·84	2·44066	603

TABLE I.—(Continued.)

Zenith Dist.	Mean Refraction	Log.	Diff.	Zenith Dist.	Mean Refraction.	Log.	Diff.
° ′	′ ″			° ′	′ ″		
78 30	4 39·75	2·44677	611	84 20	8 55·25	2·72856	1069
40	43·76	2·45295	618	30	9 8·88	2·73943	1092
50	47·88	2·45921	626	40	23·16	2·75063	1115
79 00	52·12	2·46556	635	50	38·12	2·76202	1139
10	56·47	2·47198	642	85 00	53·84	2·77367	1165
20	5 0·94	2·47848	650	10	10 10·35	2·78558	1191
30	5·54	2·48507	659	20	27·73	2·79777	1219
40	10·28	2·49176	669	30	46·03	2·81025	1248
50	15·16	2·49853	677	40	11 5·30	2·82302	1277
80 00	20·19	2·50541	688	50	25·66	2·83611	1309
10	25·36	2·51237	696	86 00	47·15	2·84951	1340
20	30·70	2·51944	707	10	12 9·88	2·86325	1374
30	36·20	2·52660	716	20	33·97	2·87735	1410
40	41·88	2·53387	727	30	59·51	2·89182	1447
50	47·74	2·54125	738	40	13 26·61	2·90666	1484
81 00	53·79	2·54874	749	50	13 55·40	2·92189	1523
10	6 0·04	2·55635	761	87 00	14 26·04	2·93754	1565
20	6·50	2·56407	772	10	14 58·71	2·95362	1608
30	13·18	2·57192	785	20	15 33·60	2·97016	1654
40	20·09	2·57989	797	30	16 10·89	2·98717	1701
50	27·26	2·58800	811	40	16 50·8	3·00466	1749
82 00	34·68	2·59624	824	50	17 33·6	3·02267	1801
10	42·37	2·60462	838	88 00	18 19·6	3·04122	1855
20	50·33	2·61313	851	10	19 9·0	3·06031	1909
30	58·59	2·62179	866	20	20 2·2	3·07998	1967
40	7 7·19	2·63062	883	30	20 59·6	3·10024	2026
50	16·13	2·63961	899	40	22 1·7	3·12113	2089
83 00	25·40	2·64875	914	50	23 8·9	3·14268	2155
10	35·05	2·65806	931	89 00	24 21·8	3·16489	2221
20	45·10	2·66755	949	10	25 40·9	3·18779	2290
30	55·58	2·67722	967	20	27 7·1	3·21140	2361
40	8 6·50	2·68708	986	30	28 40·8	3·23574	2434
50	17·90	2·69714	1006	40	30 23·2	3·26083	2509
84 00	29·80	2·70740	1026	50	32 15·0	3·28667	2584
10	8 42·24	2·71787	1047	90 00	34 17·5	3·31334	2667

TABLE II.

Mr. Ivory's Refractions continued: showing the logarithms of the corrections, on account of the state of the Thermometer and Barometer.

Thermometer.				Barometer.	
	Logarithm.		Logarithm.		Logarithm.
°		°		in.	
80	9·97237	50	0·00000	31·0	0·01424
79	9·97326	49	0·00094	30·9	0·01248
78	9·97416	48	0·00190	8	0·01143
77	9·97506	47	0·00285	7	0·01002
76	9·97596	46	0·00380	6	0·00860
75	9·97686	45	0·00476	5	0·00718
74	9·97777	44	0·00572	4	0·00575
73	9·97867	43	0·00668	3	0·00432
72	9·97958	42	0·00764	2	0·00289
71	9·98049	41	0·00861	1	0·00145
70	9·98140	40	0·00957	30·0	0·00000
69	9·98231	39	0·01053	29·9	9·99855
68	9·98323	38	0·01151	8	9·99709
67	9·98414	37	0·01248	7	9·99563
66	9·98506	36	0·01346	6	9·99417
65	9·98598	35	0·01444	5	9·99270
64	9·98690	34	0·01541	4	9·99123
63	9·98783	33	0·01640	3	9·98975
62	9·98875	32	0·01738	2	9·98826
61	9·98969	31	0·01837	1	9·98677
60	9·99061	30	0·01935	29·0	9·98528
59	9·99154	29	0·02033	28·9	9·98378
58	9·99248	28	0·02133	8	9·98227
57	9·99341	27	0·02232	7	9·98076
56	9·99434	26	·02331	6	9·97924
55	9·99529	25	0·02432	5	9·97772
54	9·99623	24	0·02531	4	9·97620
53	9·99717	23	0·02630	3	9·97466
52	9·99811	22	0·02730	2	9·97313
51	9·99906	21	0·02832	1	9·97158
50	0·00000	20	0·02933	28·0	9·97004

TABLE III.

Mr. Ivory's Refractions continued: showing the *further* quantities by which the refraction at *low* altitudes is to be corrected, on account of the state of the Thermometer and Barometer.

Zenith Distance.	T	B	Zenith Distance.	T	B
° ′	″		° ′	″	″
75 0	− 0·009		86 30	− 0·317	+ 0·51
76 0	0·012		86 40	0·345	0·56
77 0	0·015		86 50	0·376	0·62
78 0	0·018		87 0	0·410	0·68
79 0	0·023	″	87 10	0·448	0·75
80 0	0·030	+ 0·04	87 20	0·490	0·83
81 0	0·040	0·05	87 30	0·538	0·91
81 30	0·046	0·07	87 40	0·593	1·01
82 0	0·053	0·08	87 50	0·654	1·13
82 30	0·063	0·10	88 0	0·722	1·26
83 0	0·074	0·11	88 10	0·799	1·41
83 30	0·089	0·13	88 20	0·887	1·59
84 0	0·107	0.16	88 30	0·987	1·79
84 30	0·130	0·20	88 40	1·101	2·02
85 0	0·159	0·25	88 50	1·231	2·29
85 10	0·171	0·26	89 0	1·380	2·61
85 20	0·184	0·28	89 10	1·551	2·98
85 30	0·198	0·31	89 20	1·749	3·41
85 40	0·213	0·33	89 30	1·977	3·93
85 50	0·229	0·36	89 40	2·241	4·54
86 0	0·248	0·39	89 50	2·549	5·26
86 10	0·269	0·43	90 0	− 2·909	+ 6·12
86 20	− 0·292	+ 0·47			

The column marked T is to be multiplied by $(t - 50°)$; and the column marked B is to be multiplied by $(b - 30^{in}\cdot 00)$. The results are to be applied to the approximate refraction obtained by the preceding tables.

TABLE IV.

For the Equation of Equal Altitudes of the Sun.

Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.
h. m.			h. m.			h. m.		
2 0	7·7297	7·7146	3 0	7·7359	7·7015	4 0	7·7447	7·6823
2	·7298	·7143	2	·7362	·7010	2	·7451	·6815
4	·7300	·7139	4	·7364	·7005	4	·7454	·6807
6	·7302	·7136	6	·7367	·6999	6	·7458	·6800
8	·7304	·7132	8	·7369	·6993	8	·7461	·6792
10	·7305	·7128	10	·7372	·6988	10	·7464	·6784
12	·7307	·7125	12	·7374	·6982	12	·7468	·6776
14	·7309	·7121	14	·7377	·6976	14	·7472	·6768
16	·7311	·7117	16	·7380	·6970	16	·7475	·6759
18	·7313	·7113	18	·7383	·6964	18	·7479	·6751
20	·7315	·7109	20	·7386	·6958	20	·7482	·6743
22	·7317	·7105	22	·7388	·6952	22	·7486	·6734
24	·7319	·7101	24	·7391	·6946	24	·7490	·6726
26	·7321	·7097	26	·7394	·6940	26	·7494	·6717
28	·7323	·7092	28	·7397	·6934	28	·7497	·6708
30	·7325	·7088	30	·7400	·6927	30	·7501	·6700
32	·7327	·7083	32	·7403	·6921	32	·7505	·6691
34	·7329	·7079	34	·7406	·6914	34	·7509	·6682
36	·7331	·7075	36	·7409	·6908	36	·7513	·6673
38	·7333	·7070	38	·7412	·6901	38	·7517	·6663
40	·7336	·7065	40	·7415	·6894	40	·7521	·6654
42	·7338	·7061	42	·7418	·6888	42	·7525	·6645
44	·7340	·7056	44	·7421	·6881	44	·7529	·6635
46	·7342	·7051	46	·7424	·6874	46	·7533	·6626
48	·7345	·7046	48	·7428	·6867	48	·7537	·6616
50	·7347	·7041	50	·7431	·6859	50	·7541	·6606
52	·7349	·7036	52	·7434	·6852	52	·7545	·6597
54	·7352	·7031	54	·7437	·6845	54	·7549	·6587
56	·7354	·7026	56	·7441	·6838	56	·7553	·6577
58	7·7357	7·7021	58	7·7444	7·6830	58	7·7557	7·6567

Table IV.—(Continued.)

For the Equation of Equal Altitudes of the Sun.

Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.
h. m.			h. m.			h. m.		
5 0	7·7562	7.6556	6 0	7·7703	7·6198	7 0	7·7873	7·5717
2	·7566	·6546	2	.7708	·6184	2	·7879	·5699
4	·7570	·6536	4	·7713	·6170	4	·7885	·5680
6	·7575	·6525	6	·7719	·6156	6	·7891	·5661
8	·7579	·6514	8	·7724	·6142	8	·7898	·5641
10	·7583	·6504	10	·7729	·6127	10	·7904	·5622
12	·7588	·6493	12	·7735	·6113	12	·7910	·5602
14	·7592	·6482	14	·7740	·6098	14	·7916	·5582
16	·7597	·6471	16	·7745	·6083	16	·7923	·5562
18	·7601	·6460	18	·7751	·6068	18	·7929	·5542
20	·7606	·6448	20	·7756	·6053	20	·7936	·5522
22	.7610	.6437	22	·7762	·6038	22	·7942	·5501
24	·7615	·6425	24	·7767	·6023	24	·7949	·5480
26	·7620	.6414	26	·7773	·6007	26	·7955	·5459
28	·7624	·6402	28	·7779	·5991	28	·7962	·5437
30	·7629	.6390	30	·7784	·5975	30	·7969	·5416
32	·7634	·6378	32	·7790	·5959	32	·7975	·5394
34	·7638	·6366	34	·7796	·5943	34	·7982	·5372
36	·7643	·6354	36	·7801	·5927	36	·7989	·5350
38	·7648	·6342	38	·7807	·5910	38	·7995	·5327
40	·7653	·6329	40	·7813	·5894	40	·8002	·5304
42	·7658	·6317	42	·7819	·5877	42	·8009	·5281
44	·7663	·6304	44	·7825	·5860	44	·8016	·5258
46	·7668	·6291	46	·7831	·5843	46	·8023	·5234
48	·7673	·6278	48	·7836	·5825	48	·8030	·5211
50	·7678	·6265	50	·7842	·5808	50	·8037	·5186
52	·7683	·6252	52	·7848	·5790	52	·8044	·5162
54	·7688	·6239	54	·7854	·5772	54	·8051	·5137
56	·7693	·6225	56	·7860	·5754	56	·8058	·5112
58	7·7698	7·6212	58	7·7867	7·5736	58	7·8065	7·5087

Table IV.—(Continued.)

For the Equation of Equal Altitudes of the Sun.

Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.
h. m			h. m.			h. m.		
8 0	7·8072	7·5062	9 0	7·8302	7·4131	10 0	7·8567	7·2697
2	·8079	·5036	2	·8311	·4093	2	·8576	·2635
4	·8086	·5010	4	·8319	·4055	4	·8586	·2572
6	·8094	·4983	6	·8328	·4016	6	·8595	·2507
8	·8101	·4957	8	·8336	·3977	8	·8605	·2442
10	·8108	·4930	10	·8344	·8937	10	·8614	·2374
12	·8116	·4902	12	·8353	·3896	12	·8624	·2306
14	·8123	·4874	14	·8361	·3855	14	·8634	·2236
16	·8130	·4846	16	·8370	·3813	16	·8643	·2164
18	·8138	·4818	18	·8378	·3771	18	·8653	·2091
20	·8145	·4789	20	·8387	·3728	20	·8663	·2016
22	·8153	·4760	22	·8396	·3684	22	·8673	·1940
24	·8160	·4731	24	·8404	·3639	24	·8683	·1861
26	·8168	·4701	26	·8413	·3594	26	·8693	·1781
28	·8176	·4671	28	·8422	·3548	28	·8703	·1699
30	·8183	·4640	30	·8430	·3501	30	·8713	·1615
32	·8191	·4609	32	·8439	·3454	32	·8723	·1529
34	·8199	·4578	34	·8448	·3406	34	·8733	·1440
36	·8206	·4546	36	·8457	·3357	36	·8743	·1349
38	·8214	·4514	38	·8466	·3307	38	·8753	·1256
40	·8222	·4482	40	·8475	·3256	40	·8763	·1160
42	·8230	·4449	42	·8484	·3205	42	·8773	·1061
44	·8238	·4415	44	·8493	·3152	44	·8784	·0960
46	8246	·4381	46	·8502	·3099	46	·8794	·0855
48	·8254	·4347	48	·8511	·3045	48	·8804	·0748
50	·8262	·4312	50	·8520	·2989	50	·8815	·0637
52	·8270	·4277	52	·8530	·2933	52	·8825	·0522
54	·8278	·4241	54	·8539	·2876	54	·8836	·0404
56	·8286	·4205	56	·8548	·2817	56	·8846	·0282
58	7·8294	7·4168	58	7·8558	7·2758	58	7·8857	7·0156

TABLE IV.—(Continued.)

For the Equation of Equal Altitudes of the Sun.

Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.
h. m.			h. m.			h. m.		
11 0	7·8868	7.0025	12 0	7·9208	$B=0$	13 0	7.9593	— 7·0750
2	·8878	6·9889	2	.9220	— 5·5549	2	·9607	·0905
4	·8889	·9748	4	·9232	5·8641	4	·9620	·1056
6	·8900	·9602	6	·9245	6·0414	6	·9634	·1203
8	·8911	·9449	8	·9257	·1675	8	·9648	·1345
10	·8922	·9290	10	·9269	·2657	10	·9662	·1484
12	·8932	·9125	12	·9281	·3461	12	·9676	·1619
14	·8943	·8953	14	·9294	·4142	14	·9690	·1751
16	·8954	·8770	16	·9306	·4734	16	·9704	·1880
18	·8965	·8580	18	·9319	·5258	18	·9718	·2006
20	·8977	·8379	20	·9331	·5728	20	·9732	·2129
22	.8988	.8168	22	·9344	·6154	22	·9746	·2249
24	·8999	·7945	24	·9357	·6545	24	·9761	·2367
26	·9010	.7709	26	·9369	·6905	26	·9775	·2482
28	·9021	·7457	28	·9382	·7239	28	·9789	2595
30	·9033	.7189	30	·9395	·7551	30	·9804	·2706
32	·9044	·6901	32	·9408	·7843	32	·9818	·2815
34	·9055	·6591	34	·9421	·8119	34	·9833	·2922
36	·9067	·6255	36	·9433	·8380	36	·9848	·3026
38	·9078	·5889	38	·9446	·8627	38	·9862	·3129
40	·9090	·5487	40	·9460	·8863	40	·9877	·3231
42	·9102	·5041	42	·9473	·9087	42	·9892	·3330
44	·9113	·4541	44	·9486	·9302	44	·9907	·3428
46	·9125	·3973	46	·9499	·9507	46	·9922	·3524
48	·9137	·3316	48	·9512	·9705	48	·9937	·3619
50	·9148	·2536	50	·9526	6·9895	50	·9952	·3712
52	·9160	·1579	52	·9539	7·0078	52	·9967	·3804
54	·9172	6·0341	54	·9552	·0254	54	·9982	·3894
56	·9184	5·8593	56	·9566	·0425	56	7·9998	·3984
58	7·9196	5·5594	58	7·9580	— 7·0590	58	8·0013	— 7·4071

TABLE IV.—(Continued.)

For the Equation of Equal Altitudes of the Sun.

Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.
h. m.			h. m.			h. m.		
14 0	8·0028	− 7·4158	15 0	8·0521	− 7·6350	16 0	8·1082	− 7·8072
2	·0044	·4244	2	·0539	·6413	2	·1102	·8125
4	·0059	·4328	4	·0556	·6475	4	·1122	·8177
6	·0075	·4412	6	·0574	·6537	6	·1143	·8229
8	·0090	·4494	8	·0592	·6599	8	·1163	·8281
10	·0106	·4575	10	·0610	·6660	10	·1183	·8333
12	·0122	·4655	12	·0628	·6721	12	·1204	·8385
14	·0138	·4735	14	·0646	·6781	14	·1224	·8436
16	·0154	·4813	16	·0664	·6841	16	·1245	·8487
18	·0170	·4890	18	·0682	.6900	18	·1266	·8538
20	·0186	.4967	20	·0700	·6959	20	·1287	·8589
22	·0202	.5043	22	·0718	·7018	22	·1308	·8640
24	·0218	·5118	24	·0737	·7077	24	·1329	·8690
26	·0234	·5192	26	·0755	·7135	26	·1350	·8740
28	·0250	·5265	28	·0774	·7192	28	·1371	·8790
30	.0267	·5338	30	·0792	·7249	30	·1393	·8840
32	·0283	·5410	32	·0811	·7306	32	·1414	·8890
34	·0300	·5481	34	.0830	·7363	34	·1436	·8939
36	·0316	·5551	36	·0849	·7419	36	·1458	·8989
38	·0333	·5621	38	·0868	·7475	38	·1479	·9038
40	·0350	·5690	40	·0887	·7531	40	·1501	·9087
42	·0367	·5759	42	·0906	·7586	42	·1523	·9136
44	·0384	·5827	44	·0925	·7641	44	.1545	·9185
46	·0400	·5894	46	·0945	7696	46	·1568	·9234
48	·0417	·5961	48	·0964	7751	48	1590	·9282
50	·0435	·6027	50	·0983	·7805	50	·1612	·9330
52	·0452	.6092	52	·1003	·7859	52	·1635	·9379
54	·0469	·6158	54	·1023	·7912	54	·1658	·9427
56	·0486	·6222	56	·1042	·7966	56	·1680	·9475
58	8·0504	− 7·6286	58	8·1062	− 7·8019	58	8·1703	− 7·9523

Table IV.—(Continued.)

For the Equation of Equal Altitudes of the Sun.

Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.
h. m.			h. m.			h. m.		
17 0	8·1726	– 7·9571	18 0	8·2474	– 8·0969	19 0	8.3359	– 8·2354
2	·1749	·9618	2	.2501	·1015	2	·3392	·2401
4	·1773	·9666	4	·2529	·1061	4	·3424	·2448
6	·1796	·9713	6	·2556	·1107	6	·3457	·2495
8	·1819	·9761	8	·2583	·1153	8	·3490	·2542
10	·1843	·9808	10	·2611	·1199	10	·3524	·2589
12	·1867	·9855	12	·2639	·1245	12	·3557	·2637
14	·1890	·9902	14	·2667	·1291	14	·3591	·2684
16	·1914	·9949	16	·2695	·1336	16	·3625	·2732
18	·1938	7·9996	18	·2723	·1382	18	·3659	·2779
20	·1963	8·0043	20	·2752	·1428	20	·3694	2827
22	.1987	.0090	22	·2781	·1474	22	·3728	·2875
24	·2011	·0137	24	·2809	·1520	24	·3763	·2923
26	·2036	.0184	26	·2838	·1566	26	·3798	·2971
28	·2061	·0230	28	·2868	·1612	28	·3834	·3019
30	·2086	.0277	30	·2897	·1658	30	·3869	·3068
32	·2111	·0323	32	·2926	·1704	32	·3905	·3116
34	·2136	·0370	34	·2956	·1750	34	·3941	·3165
36	·2161	·0416	36	·2986	·1797	36	·3978	·3214
38	·2186	·0462	38	·3016	·1842	38	·4015	·3263
40	·2212	·0508	40	·3046	·1889	40	·4052	·3312
42	·2237	·0555	42	·3077	·1935	42	·4089	·3361
44	·2263	·0601	44	·3107	·1981	44	·4126	·3410
46	·2289	·0647	46	·3138	·2028	46	·4164	·3460
48	·2315	·0693	48	·3169	·2074	48	·4202	·3510
50	·2341	·0739	50	·3200	·2121	50	·4241	·3560
52	2367	·0785	52	·3232	·2167	52	·4279	·3610
54	·2394	·0831	54	·3263	·2214	54	·4318	·3660
56	·2420	·0877	56	·3295	·2261	56	·4357	·3711
58	8·2447	– 8·0923	58	8·3327	– 8·2307	58	8·4397	– 8·3761

Table IV.—(Continued.)

For the Equation of Equal Altitudes of the Sun.

Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.
h. m.			h. m.			h. m.		
20 0	8.4437	– 8.3812	21 0	8.5810	– 8.5466	22 0	8.7711	– 8.7560
2	.4477	.3863	2	.5863	.5527	2	.7789	.7643
4	.4518	.3915	4	.5917	.5588	4	.7868	.7727
6	.4559	.3966	6	.5971	.5650	6	.7948	.7813
8	.4600	.4018	8	.6025	.5712	8	.8030	.7899
10	.4641	.4070	10	.6081	.5775	10	.8113	.7987
12	.4683	.4122	12	.6136	.5838	12	.8198	.8076
14	.4726	.4175	14	.6193	.5902	14	.8284	.8167
16	.4768	.4227	16	.6250	.5966	16	.8372	.8259
18	.4811	.4280	18	.6308	.6031	18	.8461	.8353
20	.4854	.4334	20	.6366	.6096	20	.8553	.8448
22	.4898	.4387	22	.6426	.6162	22	.8645	.8545
24	.4942	.4441	24	.6486	.6229	24	.8740	.8644
26	.4987	.4495	26	.6546	.6296	26	.8837	.8745
28	.5032	.4549	28	.6608	.6364	28	.8935	.8847
30	.5077	.4604	30	.6670	.6433	30	.9036	.8952
32	.5123	.4659	32	.6733	.6502	32	.9139	.9058
34	.5169	.4714	34	.6796	.6572	34	.9244	.9167
36	.5215	.4770	36	.6861	.6643	36	.9351	.9278
38	.5262	.4826	38	.6927	.6715	38	.9461	.9391
40	.5310	.4882	40	.6993	.6788	40	.9574	.9507
42	.5357	.4939	42	.7060	.6860	42	.9689	.9626
44	.5406	.4996	44	.7128	.6934	44	.9807	.9747
46	.5455	.5053	46	.7197	.7009	46	8.9928	.9871
48	.5504	.5[illegible]11	48	.7268	.7085	48	9.0052	8.9999
50	.5554	.5169	50	.7339	.7162	50	.0180	9.0129
52	.5604	.5228	52	.7411	.7239	52	.0311	.0263
54	.5655	.5287	54	.7484	.7318	54	.0446	.0401
56	.5706	.5346	56	.7558	.7398	56	.0585	.0543
58	8.5758	– 8.5406	58	8.7634	– 8.7478	58	9.0729	– 9.0689

Table IV.—(Continued.)

For the Equation of Equal Altitudes of the Sun.

Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.	Interval	Log. A.	Log. B.
h. m.			h. m.			h. m.		
23 0	9·0877	− 9.0839	23 20	9·2693	− 9·2677	23 40	9.5761	− 9·5757
2	·1029	·0995	22	.2922	·2907	42	·6224	·6221
4	·1187	·1155	24	·3162	·3149	44	·6742	·6739
6	·1351	·1321	26	·3416	·3404	46	·7328	·7326
8	·1520	·1492	28	·3685	·3674	48	·8003	·8001
10	·1696	·1670	30	·3971	·3962	50	·8801	·8800
12	·1879	·1855	32	·4276	·4268	52	9·9776	9·9775
14	·2069	·2047	34	·4604	·4597	54	0·1031	0·1031
16	·2268	·2248	36	·4957	·4952	56	0·2798	0·2798
18	9·2476	− 9·2456	38	9·5341	− 9·5336	58	0·5814	− 0·5814

TABLE V.

For the Reduction to the Meridian: showing the value of

$$A = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}.$$

Sec.	0^m	1^m	2^m	3^m	4^m	5^m	6^m	7^m
	"	"	"	"	"	"	"	"
0	0·0	2·0	7·8	17·7	31·4	49·1	70·7	96·2
1	0·0	2·0	8·0	17·9	31·7	49·4	71·1	96·7
2	0·0	2·1	8·1	18·1	31·9	49·7	71·5	97·1
3	0·0	2·2	8·2	18·3	32·2	50·1	71·9	97·6
4	0·0	2·2	8·4	18·5	32·5	50·4	72·3	98·0
5	0·0	2·3	8·5	18·7	32·7	50·7	72·7	98·5
6	0·0	2·4	8·7	18·9	33·0	51·1	73·1	99·0
7	0·0	2·4	8·8	19·1	33·3	51·4	73·5	99·4
8	0·0	2·5	8·9	19·3	33·5	51·7	73·9	99·9
9	0·0	2·6	9·1	19·5	33·8	52·1	74·3	100·4
10	0·1	2·7	9·2	19·7	34·1	52 4	74·7	100·8
11	0·1	2·7	9·4	19·9	34·4	52·7	75·1	101·3
12	0·1	2·8	9·5	20·1	34·6	53·1	75·5	101·8
13	0·1	2·9	9·6	20·3	34·9	53·4	75·9	102·3
14	0·1	3·0	9·8	20·5	35·2	53·8	76·3	102·7
15	0·1	3·1	9·9	20·7	35·5	54·1	76·7	103·2
16	0·1	3·1	10·1	20·9	35·7	54·5	77·1	103·7
17	0·2	3·2	10·2	21·2	36·0	54·8	77·5	104·2
18	0·2	3·3	10·4	21·4	36·3	55·1	77·9	104·6
19	0·2	3·4	10·5	21·6	36·6	55·5	78·3	105·1
20	0·2	3·5	10·7	21·8	36·9	55·8	78·8	105·6
21	0·2	3·6	10·8	22·0	37·2	56·2	79·2	106·1
22	0·3	3·7	11·0	22·3	37·4	56·5	79·6	106·6
23	0·3	3·8	11·2	22·5	37·7	56·9	80·0	107·0
24	0·3	3·8	11·3	22·7	38·0	57·3	80·4	107·5
25	0·3	3·9	11·5	22·9	38·3	57·6	80·8	108·0
26	0·4	4·0	11·6	23·1	38·6	58·0	81·3	108·5
27	0·4	4·1	11·8	23·4	38·9	58·3	81·7	109·0
28	0·4	4·2	11·9	23·6	39·2	58·7	82·1	109.5
29	0·5	4·3	12·1	23·8	39·5	59·0	82·5	110·0

TABLE V.—(Continued.)

For the Reduction to the Meridian : showing the value of

$$A = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}.$$

Sec.	0m	1m	2m	3m	4m	5m	6m	7m
	"	"	"	"	"	"	"	"
30	0·5	4·4	12·3	24·0	39·8	59·4	83·0	110·4
31	0·5	4·5	12·4	24·3	40·1	59·8	83·4	110·9
32	0·6	4·6	12·6	24·5	40·3	60·1	83·8	111·4
33	0·6	4·7	12·8	24·7	40·6	60·5	84·2	111·9
34	0·6	4·8	12·9	25·0	40·9	60·8	84·7	112·4
35	0·7	4·9	13·1	25·2	41·2	61·2	85·1	112·9
36	0·7	5·0	13·3	25·4	41·5	61.6	85·5	113.4
37	0·7	5·1	13·4	25·7	41·8	61·9	86·0	113.9
38	0·8	5·2	13·6	25·9	42·1	62·3	86·4	114·4
39	0·8	5·3	13·8	26·2	42·5	62·7	86·8	114·9
40	0·9	5·4	14·0	26·4	42·8	63·0	87·3	115·4
41	0·9	5·6	14·1	26·6	43·1	63·4	87·7	115·9
42	1·0	5·7	14·3	26·9	43.4	63·8	88·1	116·4
43	1·0	5·8	14·5	27·1	43·7	64·2	88·6	116·9
44	1·1	5·9	14·7	27·4	44·0	64·5	89·0	117·4
45	1·1	6·0	14·8	27·6	44·3	64·9	89·5	117·9
46	1·2	6·1	15·0	27·9	44·6	65·3	89·9	118·4
47	1·2	6·2	15·2	28·1	44·9	65·7	90·3	118·9
48	1·3	6·4	15·4	28·3	45·2	66·0	90·8	119·5
49	1·3	6·5	15·6	28·6	45·5	66·4	91·2	120·0
50	1·4	6·6	15·8	28·8	45·9	66·8	91·7	120·5
51	1·4	6·7	15·9	29·1	46·2	67·2	92·1	121·0
52	1·5	6·8	16·1	29·4	46·5	67·6	92·6	121·5
53	1·5	7·0	16·3	29·6	46·8	68·0	93·0	122·0
54	1·6	7·1	16·5	29 9	47·1	68·3	93·5	122·5
55	1·6	7·2	16·7	30·1	47 5	68·7	93·9	123·1
56	1·7	7·3	16·9	30·4	47·8	69·1	94·4	123·6
57	1·8	7·5	17·1	30·6	48·1	69·5	94·8	124·1
58	1·8	7·6	17·3	30·9	48·4	69·9	95·3	124·6
59	1·9	7·7	17·5	31·1	48·8	70·3	95·7	125·1

TABLE V.—(Continued.)

For the Reduction to the Meridian: showing the value of

$$A = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}.$$

Sec.	8^m	9^m	10^m	11^m	12^m	13^m	14^m
	"	"	"	"	"	"	"
0	125·7	159·0	196·3	237·5	282·7	331·8	384·7
1	126·2	159·6	197·0	238·3	283·5	332·6	385·6
2	126·7	160·2	197·6	239·0	284·2	333·4	386·6
3	127·2	160·8	198·3	239·7	285·0	334·3	387·5
4	127·8	161·4	198·9	240·4	285·8	335·2	388·4
5	128·3	162·0	199·6	241·2	286·6	336·0	389·3
6	128·8	162·6	200·3	241·9	287·4	336·9	390·2
7	129·3	163·2	200·9	242·6	288·2	337·7	391·1
8	129·9	163·8	201·6	243·3	289·0	338·6	392·1
9	130·4	164·4	202·2	244·1	289·8	339·4	393·0
10	131·0	165·0	202·9	244·8	290·6	340·3	393·9
11	131·5	165·6	203·6	245·5	291·4	341·2	394·8
12	132·0	166·2	204·2	246·3	292·2	342·0	395·8
13	132·6	166·8	204·9	247·0	293·0	342·9	396·7
14	133·1	167·4	205·6	247·7	293·8	343·7	397·6
15	133·6	168·0	206·3	248·5	294·6	344·6	398·6
16	134·2	168·6	206·9	249·2	295·4	345·5	399·5
17	134·7	169·2	207·6	249·9	296·2	346·4	400·5
18	135·3	169·8	208·3	250·7	297·0	347·2	401·4
19	135·8	170·4	208·9	251·4	297·8	348·1	402·3
20	136·3	171·0	209·6	252·2	298·6	349·0	403·3
21	136·9	171·6	210·3	253·0	299·4	349·8	404·2
22	137·4	172·2	211·0	253·6	300·2	350·7	405·1
23	138·0	172·9	211·7	254·4	301·0	351·6	406·0
24	138·5	173·5	212·3	255·1	301·8	352·5	407·0
25	139·1	174·1	213·0	255·9	302·6	353·3	408·0
26	139·6	174 7	213·7	256·6	303·5	354·2	408·9
27	140·2	175·3	214·4	257·4	304·3	355·1	409·9
28	140·7	175·9	215·1	258·1	305·1	356·0	410·8
29	141·3	176·6	215·8	258·9	305·9	356·9	411·7

TABLE V.—(Continued.)

For the Reduction to the Meridian: showing the value of

$$A = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}.$$

Sec.	8^m	9^m	10^m	11^m	12^m	13^m	14^m
	"	"	"	"	"	"	"
30	141·8	177·2	216·4	259·6	306·7	357·7	412·7
31	142·4	177·8	217·1	260·4	307·5	358·6	413.6
32	143·0	178·4	217·8	261·1	308·4	359·5	414·6
33	143.5	179·0	218·5	261·9	309·2	360·4	415·5
34	144·1	179·7	219·2	262·6	310·0	361.3	416·5
35	144·6	180·3	219·9	263·4	310·8	362·2	417·5
36	145·2	180·9	220·6	264·1	311·6	363·1	418·4
37	145·8	181·6	221·3	264·9	312·5	364·0	419·4
38	146·3	182·2	222·0	265·7	313.3	364·8	420·3
39	146·9	182·8	222·7	266·4	314·1	365·7	421·3
40	147·5	183·5	223·4	267·2	315·0	366·6	422·2
41	148·0	184·1	224·1	267·9	315·8	367·5	423·2
42	148·6	184·7	224·8	268·7	316·6	368·4	424·2
43	149·2	185·4	225·5	269·5	317·4	369·3	425·1
44	149·7	186·0	226·2	270·3	318·3	370·2	426·1
45	150·3	186·6	226·9	271·0	319·1	371·1	427·0
46	150·9	187·3	227·6	271·8	319·9	372·0	428·0
47	151·5	187·9	228·3	272·6	320·8	372·9	429·0
48	152·0	188·5	229·0	273·3	321·6	373·8	429·9
49	152·6	189·2	229·7	274·1	322·4	374·7	430·9
50	153·2	189·8	230·4	274·9	323·3	375·6	431·9
51	153·8	190·5	231·1	275·6	324·1	376·5	432·8
52	154·4	191·1	231·8	276·4	325·0	377·4	433·8
53	154·9	191·8	232·5	277·2	325·8	378·3	434·8
54	155·5	192·4	233·2	278·0	326·7	379·3	435·8
55	156·1	193·1	234·0	278·8	327·5	380·2	436·7
56	156·7	193·7	234·7	279·5	328·4	381·1	437·7
57	157·3	194·4	235·4	280·3	329·2	382·0	438·7
58	157·8	195·0	236·1	281·1	330·0	382·9	439·7
59	158·4	195·7	236·8	281·9	330·9	383·8	440·6

TABLE V.—(Continued.)

For the Reduction to the Meridian: showing the value of

$$A = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}.$$

Sec.	15m	16m	17m	18m	19m	20m	21m
	"	"	"	"	"	"	"
0	441·6	502·5	567·2	635·9	708·4	784·9	865·3
1	442·6	503·5	568·3	637·0	709·7	786·2	866·6
2	443·6	504·6	569·4	638·2	710·9	787·5	868·0
3	444·6	505·6	570·5	639·4	712·1	788·8	869·4
4	445·6	506·7	571·6	640·6	713·4	790·1	870·8
5	446·5	507·7	572·8	641·7	714·6	791·4	872·1
6	447·5	508·8	573·9	642·9	715·9	792·7	873·5
7	448·5	509·8	575·0	644·1	717·1	794·0	874·9
8	449·5	510·9	576·1	645·3	718·4	795·4	876·3
9	450·5	511·9	577·2	646·5	719·6	796·7	877·6
10	451·5	513·0	578·4	647·7	720·9	798·0	879·0
11	452·5	514·0	579·5	648·9	722·1	799·3	880·4
12	453·5	515·1	580·6	650·0	723·4	800·7	881·8
13	454·5	516·1	581·7	651·2	724·6	802·0	883·2
14	455·5	517·2	582·9	652·4	725·9	803·3	884·6
15	456·5	518·3	584·0	653·6	727·2	804·6	886·0
16	457·5	519·3	585·1	654·8	728·4	806·0	887·4
17	458·5	520·4	586·2	656·0	729·7	807·3	888·8
18	459·5	521·5	587·4	657·2	730·9	808·6	890·2
19	460·5	522·5	588·5	658·4	732·2	809·9	891·6
20	461·5	523·6	589·6	659·6	733·5	811·3	893·0
21	462·5	524·6	590·8	660·8	734·7	812·6	894·4
22	463·5	525·7	591·9	662·0	736·0	813·9	895·8
23	464·5	526·8	593·0	663·2	737·3	815·2	897·2
24	465·5	527·9	594·2	664·4	738·5	816·6	898·6
25	466·5	528·9	595·3	665·6	739·8	817·9	900·0
26	467·5	530·0	596·5	666·8	741·1	819·2	901·4
27	468·5	531·1	597·6	668·0	742·3	820·5	902·8
28	469·5	532·2	598·7	669·2	743·6	821·9	904·2
29	470·5	533·2	599·9	670·4	744·9	823·2	905·6

TABLE V.—(Continued.)

For the Reduction to the Meridian: showing the value of

$$A = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}.$$

Sec.	15^m	16^m	17^m	18^m	19^m	20^m	21^m
	"	"	"	"	"	"	"
30	471·5	534·3	601·0	671·6	746·2	824 6	907·0
31	472 6	535·4	602·2	672·8	747·4	825·9	908·4
32	473·6	536·5	603·3	674·1	748·7	827·3	909·8
33	474·6	537·6	604·5	675·3	750·0	828·6	911·2
34	475·6	538·7	605·6	676·5	751·3	829·9	912·6
35	476·6	539·7	606·8	677·7	752·6	831·2	914·0
36	477·6	540·8	607·9	678·9	753·8	832·6	915·5
37	478·7	541·9	609·1	680·1	755·1	833·9	916·9
38	479·7	543·0	610·2	681·3	756·4	835·3	918·3
39	480·7	544·1	611·4	682·6	757·7	836·6	919·7
40	481·7	545·2	612·5	683·8	759·0	838·0	921·1
41	482·8	546·3	613·7	685·0	760·2	839·3	922·5
42	483·8	547·4	614·8	686·2	761·5	840·7	923·9
43	484·8	548·4	616·0	687·4	762·8	842·0	925·3
44	485·8	549·5	617·2	688·7	764·1	843·4	926·8
45	486·9	550·6	618·3	689·9	765·4	844·7	928·2
46	487·9	551·7	619·5	691·1	766·7	846·1	929·6
47	488·9	552·8	620·6	692·4	768·0	847·5	931·0
48	490·0	553·9	621·8	693·6	769·3	848·9	932·4
49	491·0	555·0	623·0	694·8	770·6	850·2	933·8
50	492·0	556·1	624·1	696·0	771·9	851·6	935·2
51	493·1	557·2	625·3	697·3	773·1	852·9	936·6
52	494·1	558·3	626·5	698·5	774·5	854·3	938·1
53	495 2	559·4	627·6	699·7	775·8	855·7	939·5
54	496·2	560·5	628·8	701·0	777·1	857·1	940 ·9
55	497·2	561·6	630·0	702·2	778·4	858·4	942·3
56	498·3	562 7	631·2	703·5	779·7	859·8	943·8
57	499·3	563·9	632·3	704·7	781·0	861·1	945·2
58	500·3	565·0	633·5	705·9	782·3	862·5	946·6
59	501·4	566·1	634·7	707·[illegible]	783·6	863·9	948·1

TABLE V.—(Continued.)

For the Reduction to the Meridian: showing the value of

$$A = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}.$$

Sec.	22ᵐ	23ᵐ	24ᵐ	25ᵐ	26ᵐ	27ᵐ	28ᵐ
	″	″	″	″	″	″	″
0	949·6	1037·8	1129·9	1225·9	1325·9	1429·7	1537·5
1	951·0	1039·3	1131·4	1227·5	1327·6	1431·4	1539·3
2	952·4	1040·8	1133·0	1229·2	1329·3	1433·2	1541·1
3	953·8	1042·3	1134·6	1230·8	1331·0	1434·9	1542·9
4	955·3	1043·8	1136·2	1232·5	1332·7	1436·7	1544·8
5	956·7	1045·3	1137·8	1234·1	1334·4	1438·5	1546·6
6	958·2	1046·8	1139·3	1235·7	1336·1	1440·3	1548·4
7	959·6	1048·3	1140·9	1237·3	1337·8	1442·1	1550·2
8	961·1	1049·8	1142·5	1239·0	1339·5	1443·9	1552·1
9	962·5	1051·3	1144·0	1240·6	1341·2	1445·6	1553·9
10	963·9	1052·8	1145·6	1242·3	1342·9	1447·4	1555·8
11	965·4	1054·3	1147·2	1243·9	1344·6	1449·2	1557·6
12	966·9	1055·9	1148·8	1245·6	1346·3	1451·0	1559·5
13	968·3	1057·4	1150·4	1247·2	1348·0	1452·8	1561·3
14	969·8	1058·9	1152·0	1248·9	1349·7	1454·5	1563·2
15	971·2	1060·4	1153·6	1250·5	1351·4	1456·3	1565·0
16	972·7	1062·0	1155·2	1252·2	1353·2	1458·1	1566·9
17	974·1	1063·5	1156·8	1253·8	1354·9	1459·9	1568·7
18	975·5	1065·0	1158·3	1255·5	1356·6	1461·6	1570·5
19	977·0	1066·5	1159·9	1257·1	1358·3	1463·4	1572·4
20	978·5	1068·1	1161·5	1258·8	1360·1	1465·2	1574·3
21	979·9	1069·6	1163·1	1260·4	1361·8	1466·9	1576·1
22	981·4	1071·1	1164·7	1262·1	1363·5	1468·7	1578·0
23	982·9	1072·6	1166·3	1263·7	1365·2	1470·5	1579·8
24	984·4	1074·2	1167·9	1265·4	1367·0	1472·3	1581·7
25	985·8	1075·7	1169·5	1267·0	1368·7	1474·0	1583·5
26	987·3	1077·2	1171·1	1268·7	1370·4	1475·9	1585·3
27	988·8	1078·7	1172·7	1270·3	1372·1	1477·7	1587·2
28	990·3	1080·3	1174·3	1272·	1373·9	1479·5	1589·1
29	991·8	1081·8	1175·9	1273·7	1375·6	1481·3	1590·9

Table V.—(Continued.)

For the Reduction to the Meridian: showing the value of

$$A = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}.$$

Sec.	22^m	23^m	24^m	25^m	26^m	27^m	28^m
	″	″	″	″	″	″	″
30	993·2	1083·3	1177·5	1275·4	1377·4	1483·1	1592·7
31	994·7	1084·8	1179·1	1277·1	1379·0	1484·9	1594·6
32	996·2	1086·4	1180·7	1278·8	1380·8	1486·7	1596·5
33	997·6	1087·9	1182·3	1280·4	1382·5	1488·5	1598·3
34	999·1	1089·5	1183·9	1282·1	1384·2	1490·3	1600·2
35	1000·6	1091·0	1185·5	1283·8	1385·9	1492·1	1602·1
36	1002·1	1092·6	1187·1	1285·5	1387·7	1493·9	1604·0
37	1003·5	1094·1	1188·7	1287·1	1389·4	1495·7	1605·9
38	1005·0	1095·7	1190·3	1288·8	1391·2	1497·5	1607·7
39	1006·5	1097·2	1191·9	1290·5	1392·9	1499·3	1609·6
40	1008·0	1098·8	1193·5	1292·2	1394·7	1501·1	1611·5
41	1009·4	1100·3	1195·1	1293·8	1396·4	1502·9	1613·3
42	1010·9	1101·9	1196·7	1295·5	1398·2	1504·7	1615·2
43	1012·4	1103·4	1198·3	1297·2	1399·9	1506·5	1617·1
44	1013·9	1105·0	1199·9	1298·9	1401·7	1508·4	1619·0
45	1015·4	1106·5	1201·5	1300·5	1403·4	1510·2	1620·8
46	1016·9	1108·1	1203·1	1302·2	1405·2	1512·0	1622·7
47	1018·4	1109·6	1204·7	1303·9	1406·9	1513·8	1624·6
48	1019·9	1111·2	1206·4	1305·6	1408·7	1515·6	1626·5
49	1021·4	1112·7	1208·0	1307·3	1410·4	1517·4	1628·3
50	1022·8	1114·3	1209·6	1309·0	1412·2	1519·2	1630·2
51	1024·3	1115·8	1211·2	1310·7	1413·9	1521·0	1632·1
52	1025·8	1117·4	1212·9	1312·4	1415·7	1522·9	1634·0
53	1027·3	1118·9	1214·5	1314·1	1417·4	1524·7	1635·9
54	1028·8	1120·5	1216·1	1315·7	1419·2	1526·5	1637·7
55	1030·3	1122·0	1217·7	1317·4	1420·9	1528·3	1639·6
56	1031·8	1123·6	1219·4	1319·1	1422·7	1530·2	1641·5
57	1033·3	1125·1	1221·0	1320·8	1424·4	1532·0	1643·3
58	1034·8	1126·7	1222·6	1322·5	1426·2	1533·8	1645·2
59	1036·3	1128·3	1224·2	1324·2	1427·9	1535·6	1647·1

TABLE V.—(Continued.)

For the Reduction to the Meridian: showing the value of

$$A = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}.$$

Sec.	29ᵐ	30ᵐ	31ᵐ	32ᵐ	33ᵐ	34ᵐ	35ᵐ
	"	"	"	"	"	"	"
0	1649·0	1764·6	1884·0	2007·4	2134·6	2265·6	2400·6
1	1650·9	1766·6	1886·0	2009·4	2136·8	2267·8	2402·9
2	1652·8	1768·5	1888·0	2011·5	2138·9	2270·0	2405·2
3	1654·7	1770·5	1890·0	2013·6	2141·1	2272·2	2407·5
4	1656·6	1772·4	1892·1	2015·7	2143·2	2274·5	2409·8
5	1658·5	1774·4	1894·1	2017·8	2145·3	2276·7	2412·0
6	1660·4	1776·3	1896·1	2019·9	2147·5	2278·9	2414·3
7	1662·3	1778·3	1898·1	2022·0	2149·7	2281·2	2416·6
8	1664·2	1780·3	1900·2	2024·1	2151·8	2283·4	2418·9
9	1666·1	1782·3	1902·2	2026·2	2153·9	2285·6	2421·2
10	1668·0	1784·2	1904·3	2028·3	2156·1	2287·8	2423·5
11	1669·9	1786·2	1906·3	2030·5	2158·3	2290·0	2425·8
12	1671·9	1788·2	1908·4	2032·5	2160·5	2292·3	2428·1
13	1673·8	1790·1	1910·4	2034·6	2162·6	2294·5	2430·4
14	1675·7	1792·1	1912·4	2036·7	2164·8	2296·8	2432·7
15	1677·6	1794·1	1914·4	2038·8	2166·9	2299·0	2435·0
16	1679·5	1796·1	1916·5	2040·9	2169·1	2301·3	2437·3
17	1681·4	1798·1	1918·5	2043·0	2171·2	2303·6	2439·6
18	1683·3	1800·0	1920·6	2045·1	2173·4	2305·8	2441·9
19	1685·2	1802·0	1922·6	2047·2	2175·6	2308·0	2444·2
20	1687·2	1804·0	1924·7	2049·3	2177·8	2310·2	2446·5
21	1689·1	1805·9	1926·7	2051·4	2179·9	2312·4	2448·8
22	1691·0	1807·9	1928·8	2053·5	2182·1	2314·7	2451·1
23	1692·9	1809·9	1930·8	2055·7	2184·3	2316·9	2453·4
24	1694·8	1811·9	1932·9	2057·8	2186·5	2319·2	2455·7
25	1696·7	1813·9	1935·0	2059·9	2188·6	2321·5	2458·0
26	1698·6	1815·8	1937·0	2062·0	2190·8	2323·7	2460·3
27	1700·5	1817·8	1939·0	2064·1	2193·0	2325·9	2462·6
28	1702·5	1819·8	1941·1	2066·2	2195·2	2328·2	2464·9
29	1704·4	1821·8	1943·1	2068·3	2197·3	2330·4	2467·2

Table V.—(Continued.)

For the Reduction to the Meridian: showing the value of

$$A = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''}.$$

Sec.	29^m	30^m	31^m	32^m	33^m	34^m	35^m
	"	"	"	"	"	"	"
30	1706·3	1823·8	1945·2	2070·4	2199·5	2332·7	2469·5
31	1708·2	1825·8	1947·2	2072·6	2201·7	2334·9	2471·8
32	1710·2	1827·8	1949·3	2074·7	2203·9	2337·2	2474·2
33	1712·1	1829·8	1951·3	2076·8	2206·1	2339·4	2476·5
34	1714·0	1831·8	1953·4	2078·9	2208·3	2341·7	2478·8
35	1715·9	1833·8	1955·5	2081·0	2210·5	2343·9	2481·1
36	1717·9	1835·8	1957·6	2083·2	2212·7	2346·2	2483·5
37	1719·8	1837·8	1959·6	2085·3	2214·9	2348·5	2485·8
38	1721·7	1839·8	1961·7	2087·4	2217·1	2350·7	2488·1
39	1723·6	1841·8	1963·7	2089·6	2219·3	2353·0	2490·4
40	1725·6	1843·8	1965·8	2091·7	2221·5	2355·2	2492·8
41	1727·5	1845·8	1967·8	2093·8	2223·7	2357·5	2495·1
42	1729·5	1847·8	1969·9	2095·9	2225·9	2359·7	2497·4
43	1731·5	1849·8	1972·0	2098·0	2228·1	2361·9	2499·7
44	1733·4	1851·8	1974·1	2100·2	2230·3	2364·2	2502·1
45	1735·3	1853·8	1976·1	2102·3	2232·5	2366·4	2504·4
46	1737·2	1855·8	1978·2	2104·5	2234·7	2368·7	2506·7
47	1739·2	1857·8	1980·3	2106·6	2236·9	2371·0	2509·0
48	1741·2	1859·8	1982·4	2108·8	2239·1	2373·3	2511·4
49	1743·1	1861·8	1984·4	2110·9	2241·3	2375·5	2513·7
50	1745·1	1863·8	1986·5	2113·1	2243·5	2377·8	2516·1
51	1747·0	1865·8	1988·6	2115·2	2245·7	2380·1	2518·4
52	1749·0	1867·8	1990·7	2117·4	2247·9	2382·4	2520·8
53	1750·9	1869·8	1992·7	2119·6	2250·1	2384·6	2523·1
54	1752·9	1871·8	1994·8	2121·7	2252·3	2386·9	2525·4
55	1754·8	1873·8	1996·9	2123·8	2254·5	2389·2	2527·7
56	1756·8	1875·9	1999·0	2126·0	2256·7	2391·5	2530·1
57	1758·7	1877·9	2001·0	2128·1	2258·9	2393·7	2532·4
58	1760·7	1879·9	2003·1	2130·3	2261·1	2396·0	2534·8
59	1762·6	1882·0	2005·3	2132·4	2263·4	2398·3	2537·1

TABLE VI.

For the second part of the Reduction to the Meridian: showing the value of

$$B = \frac{2 \sin^4 \frac{1}{2} P}{\sin 1''}.$$

Minutes	0^s	10^s	20^s	30^s	40^s	50^s
	''	''	''	''	''	''
5	0·01	0·01	0·01	0·01	0·01	0·01
6	0·01	0·01	0·01	0·02	0·02	0·02
7	0·02	0·02	0·03	0·03	0·03	0·04
8	0·04	0·04	0·05	0·05	0·05	0·06
9	0·06	0·07	0·08	0·08	0·08	0·09
10	0·09	0·10	0·11	0·11	0·12	0·13
11	0·14	0·15	0·15	0·16	0·17	0·18
12	0·19	0·20	0·22	0·23	0·24	0·25
13	0·27	0·28	0·30	0·31	0·33	0·34
14	0·36	0·38	0·39	0·41	0·43	0·45
15	0·47	0·49	0·52	0·54	0·56	0·59
16	0·61	0·64	0·67	0·69	0·72	0·75
17	0·78	0·81	0·84	0·88	0·91	0·95
18	0·98	1·02	1·06	1·09	1·13	1·18
19	1·22	1·26	1·30	1·35	1·40	1·44
20	1·49	1·54	1·60	1·65	1·70	1·76
21	1·82	1·87	1·93	1·99	2·06	2·12
22	2·19	2·25	2·32	2·39	2·46	2·54
23	2·61	2·69	2·77	2·85	2·93	3·01
24	3·10	3·18	3·27	3·36	3·45	3·55
25	3·64	3·74	3·84	3·94	4·05	4·15
26	4·26	4·37	4·48	4·60	4·72	4·83
27	4·96	5·08	5·20	5·33	5·46	5·60
28	5·73	5·87	6·01	6·15	6·30	6·44
29	6·59	6·75	6·90	7·06	7·22	7·38
30	7·55	7·72	7·89	8·06	8·24	8·42
31	8·61	8·79	8·98	9·17	9·37	9·57
32	9·77	9·97	10·18	10·39	10·61	10·82
33	11·04	11·27	11·50	11·73	11·96	12·20
34	12·44	12·69	12·94	13·19	13·45	13·71
35	13·97	14·24	14·51	14·78	15·06	15·35

TRIGONOMETRICAL FORMULÆ.

I. Equivalent expressions for sin x.

1. $\cos x \,.\, \tan x$.

2. $\dfrac{\cos x}{\cot x}$.

3. $\sqrt{1 - \cos^2 x}$.

4. $\dfrac{1}{\sqrt{1 + \cot^2 x}}$.

5. $\dfrac{\tan x}{\sqrt{1 + \tan^2 x}}$.

6. $2 \sin \frac{1}{2} x \,.\, \cos \frac{1}{2} x$.

7. $\sqrt{\dfrac{1 - \cos 2x}{2}}$.

8. $\dfrac{2 \tan \frac{1}{2} x}{1 + \tan^2 \frac{1}{2} x}$.

9. $\dfrac{2}{\cot \frac{1}{2} x + \tan \frac{1}{2} x}$.

10. $\dfrac{\sin (30^\circ + x) - \sin (30^\circ - x)}{\sqrt{3}}$.

11. $2 \sin^2 (45^\circ + \frac{1}{2} x) - 1$.

12. $1 - 2 \sin^2 (45^\circ - \frac{1}{2} x)$.

13. $\dfrac{1 - \tan^2 (45^\circ - \frac{1}{2} x)}{1 + \tan^2 (45^\circ - \frac{1}{2} x)}$.

14. $\dfrac{\tan (45^\circ + \frac{1}{2} x) - \tan (45^\circ - \frac{1}{2} x)}{\tan (45^\circ + \frac{1}{2} x) + \tan (45^\circ - \frac{1}{2} x)}$.

15. $\sin (60^\circ + x) - \sin (60^\circ - x)$.

16. $\dfrac{1}{\text{cosecant } x}$.

II. Equivalent expressions for cos x.

1. $\dfrac{\sin x}{\tan x}$

2. $\sin x \,.\, \cot x.$

3. $\sqrt{1 - \sin^2 x}.$

4. $\dfrac{1}{\sqrt{1 + \tan^2 x}}.$

5. $\dfrac{\cot x}{\sqrt{1 + \cot^2 x}}.$

6. $\cos^2 \frac{1}{2} x - \sin^2 \frac{1}{2} x.$

7. $1 - 2 \sin^2 \frac{1}{2} x.$

8. $2 \cos^2 \frac{1}{2} x - 1.$

9. $\sqrt{\dfrac{1 + \cos 2x}{2}}.$

10. $\dfrac{1 - \tan^2 \frac{1}{2} x}{1 + \tan^2 \frac{1}{2} x}.$

11. $\dfrac{\cot \frac{1}{2} x - \tan \frac{1}{2} x}{\cot \frac{1}{2} x + \tan \frac{1}{2} x}.$

12. $\dfrac{1}{1 + \tan x \,.\, \tan \frac{1}{2} x}.$

13. $\dfrac{2}{\tan (45° + \frac{1}{2} x) + \cot (45° + \frac{1}{2} x)}.$

14. $2 \cos (45° + \frac{1}{2} x) \cos (45° - \frac{1}{2} x).$

15. $\cos (60° + x) + \cos (60° - x).$

16. $\dfrac{1}{\text{secant } x}.$

III. Equivalent expressions for $\tan x$.

1. $\dfrac{\sin x}{\cos x}$.

2. $\dfrac{1}{\cot x}$.

3. $\sqrt{\dfrac{1}{\cos^2 x} - 1}$.

4. $\dfrac{\sin x}{\sqrt{1 - \sin^2 x}}$.

5. $\dfrac{\sqrt{1 - \cos^2 x}}{\cos x}$.

6. $\dfrac{2 \tan \frac{1}{2} x}{1 - \tan^2 \frac{1}{2} x}$.

7. $\dfrac{2 \cot \frac{1}{2} x}{\cot^2 \frac{1}{2} x - 1}$.

8. $\dfrac{2}{\cot \frac{1}{2} x - \tan \frac{1}{2} x}$.

9. $\cot x - 2 \cot 2 x$.

10. $\dfrac{1 - \cos 2 x}{\sin 2 x}$.

11. $\dfrac{\sin 2 x}{1 + \cos 2 x}$.

12. $\sqrt{\dfrac{1 - \cos 2 x}{1 + \cos 2 x}}$.

13. $\dfrac{\tan (45° + \frac{1}{2} x) - \tan (45° - \frac{1}{2} x)}{2}$.

IV. Relative to two arcs A and B.

1. $\sin(A+B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B.$

2. $\sin(A-B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B.$

3. $\cos(A+B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B.$

4. $\cos(A-B) = \cos A \,.\, \cos B + \sin A \,.\, \sin B.$

5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \,.\, \tan B}.$

6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \,.\, \tan B}.$

7. $\sin(45^\circ \pm B)$
8. $\cos(45^\circ \mp B)$

$$\left.\begin{array}{l}\sin(45^\circ \pm B)\\ \cos(45^\circ \mp B)\end{array}\right\} = \frac{\cos B \pm \sin B}{\sqrt{2}}.$$

9. $\tan(45^\circ \pm B) = \dfrac{1 \pm \tan B}{1 \mp \tan B}.$

10. $\tan^2(45^\circ \pm \frac{1}{2} B) = \dfrac{1 \pm \sin B}{1 \mp \sin B}.$

11. $\tan(45^\circ \pm \frac{1}{2} B) = \dfrac{1 \pm \sin B}{\cos B} = \dfrac{\cos B}{1 \mp \sin B}.$

12. $\dfrac{\sin(A+B)}{\sin(A-B)} = \dfrac{\tan A + \tan B}{\tan A - \tan B} = \dfrac{\cot B + \cot A}{\cot B - \cot A}$

13. $\dfrac{\cos(A+B)}{\cos(A-B)} = \dfrac{\cot B - \tan A}{\cot B + \tan A} = \dfrac{\cot A - \tan B}{\cot A + \tan B}.$

14. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$

15. $\dfrac{\cos B + \cos A}{\cos B - \cos A} = \dfrac{\cot \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$

[*continued.*

IV. *continued.* Relative to two arcs A and B.

16. $\sin A \cdot \cos B \quad = \frac{1}{2} \sin (A + B) + \frac{1}{2} \sin (A - B)$.

17. $\cos A \cdot \sin B \quad = \frac{1}{2} \sin (A + B) - \frac{1}{2} \sin (A - B)$.

18. $\sin A \cdot \sin B \quad = \frac{1}{2} \cos (A - B) - \frac{1}{2} \cos (A + B)$.

19. $\cos A \cdot \cos B \quad = \frac{1}{2} \cos (A + B) + \frac{1}{2} \cos (A - B)$.

20. $\sin A + \sin B \quad = 2 \sin \frac{1}{2} (A + B) \cdot \cos \frac{1}{2} (A - B)$.

21. $\cos A + \cos B \quad = 2 \cos \frac{1}{2} (A + B) \cdot \cos \frac{1}{2} (A - B)$.

22. $\tan A + \tan B \quad = \dfrac{\sin (A + B)}{\cos A \cdot \cos B}$.

23. $\cot A + \cot B \quad = \dfrac{\sin (A + B)}{\sin A \cdot \sin B}$.

24. $\sin A - \sin B \quad = 2 \sin \frac{1}{2} (A - B) \cdot \cos \frac{1}{2} (A + B)$.

25. $\cos B - \cos A \quad = 2 \sin \frac{1}{2} (A - B) \cdot \sin \frac{1}{2} (A + B)$.

26. $\tan A - \tan B \quad = \dfrac{\sin (A - B)}{\cos A \cdot \cos B}$.

27. $\cot B - \cot A \quad = \dfrac{\sin (A - B)}{\sin A \cdot \sin B}$.

28. $\sin^2 A - \sin^2 B$
29. $\cos^2 B - \cos^2 A$ $\left.\right\} = \sin (A - B) \cdot \sin (A + B)$.

30. $\cos^2 A - \sin^2 B \quad = \cos (A - B) \cdot \cos (A + B)$.

31. $\tan^2 A - \tan^2 B \quad = \dfrac{\sin (A - B) \cdot \sin (A + B)}{\cos^2 A \cdot \cos^2 B}$.

32. $\cot^2 B - \cot^2 A \quad = \dfrac{\sin (A - B) \cdot \sin (A + B)}{\sin^2 A \cdot \sin^2 B}$.

V. Differences of trigonometrical lines.

1. $\Delta \sin x = + 2 \sin \tfrac{1}{2} \Delta x \,.\, \cos (x + \tfrac{1}{2} \Delta x).$

2. $\Delta \cos x = - 2 \sin \tfrac{1}{2} \Delta x \,.\, \sin (x + \tfrac{1}{2} \Delta x).$

3. $\Delta \tan x = + \dfrac{\sin \Delta x}{\cos x \,.\, \cos (x + \Delta x)}.$

4. $\Delta \cot x = - \dfrac{\sin \Delta x}{\sin x \,.\, \sin (x + \Delta x)}.$

5. $\Delta \sin^2 x = + \sin \Delta x \,.\, \sin (2x + \Delta x).$

6. $\Delta \cos^2 x = - \sin \Delta x \,.\, \sin (2x + \Delta x).$

7. $\Delta \tan^2 x = + \dfrac{\sin \Delta x \,.\, \sin (2x + \Delta x)}{\cos^2 x \,.\, \cos^2 (x + \Delta x)}.$

8. $\Delta \cot^2 x = - \dfrac{\sin \Delta x \,.\, \sin (2x + \Delta x)}{\sin^2 x \,.\, \sin^2 (x + \Delta x)}$

VI. Differentials of trigonometrical lines.

1. $d \sin x = + d x \,.\, \cos x.$

2. $d \cos x = - d x \,.\, \sin x.$

3. $d \tan x = + \dfrac{d x}{\cos^2 x}.$

4. $d \cot x = - \dfrac{d x}{\sin^2 x}.$

5. $d \sin^2 x = + 2 d x \,.\, \sin x \,.\, \cos x.$

6. $d \cos^2 x = - 2 d x \,.\, \sin x \,.\, \cos x.$

7. $d \tan^2 x = + \dfrac{2 d x \,.\, \tan x}{\cos^2 x}.$

8. $d \cot^2 x = - \dfrac{2 d x \,.\, \cot x}{\sin^2 x}.$

VII. General analytical expressions for the sides and angles of any spherical triangle.

1. $\cos S = \cos A \,.\, \sin S' \,.\, \sin S'' + \cos S' \,.\, \cos S''$

2. $\cos S' = \cos A' \,.\, \sin S'' \,.\, \sin S + \cos S'' \,.\, \cos S.$

3. $\cos S'' = \cos A'' \,.\, \sin S \,.\, \sin S' + \cos S \,.\, \cos S'.$

4. $\cos A = \cos S \,.\, \sin A' \,.\, \sin A'' - \cos A' \,.\, \cos A''.$

5. $\cos A' = \cos S' \,.\, \sin A'' \,.\, \sin A - \cos A'' \,.\, \cos A.$

6. $\cos A'' = \cos S'' \,.\, \sin A \,.\, \sin A' - \cos A \,.\, \cos A'.$

7. $\cos S \,.\, \cos A' = \cot S'' \,.\, \sin S - \sin A' \,.\, \cot A''.$

8. $\cos S' \,.\, \cos A'' = \cot S \,.\, \sin S' - \sin A'' \,.\, \cot A.$

9. $\cos S'' \,.\, \cos A = \cot S' \,.\, \sin S'' - \sin A \,.\, \cot A'.$

10. $\dfrac{\sin A}{\sin S} = \dfrac{\sin A'}{\sin S'} = \dfrac{\sin A''}{\sin S''}.$

11. $\sin \tfrac{1}{2}(S' + S) : \sin \tfrac{1}{2}(S' - S) :: \cot \tfrac{1}{2} A'' : \tan \tfrac{1}{2}(A' - A).$

12. $\cos \tfrac{1}{2}(S' + S) : \cos \tfrac{1}{2}(S' - S) :: \cot \tfrac{1}{2} A'' : \tan \tfrac{1}{2}(A' + A).$

13. $\sin \tfrac{1}{2}(A' + A) : \sin \tfrac{1}{2}(A' - A) :: \tan \tfrac{1}{2} S'' : \tan \tfrac{1}{2}(S' - S).$

14. $\cos \tfrac{1}{2}(A' + A) : \cos \tfrac{1}{2}(A' - A) :: \tan \tfrac{1}{2} S'' : \tan \tfrac{1}{2}(S' + S).$

In these formulæ A, A', A'', denote the several *angles* of the triangle; and S, S', S'', the *sides* opposite those angles respectively. For the more convenient computation of the formulæ Nos. 1–9, certain auxiliary angles are introduced, which will be alluded to in the formulæ for the solution of the several cases of oblique-angled spherical triangles.

VIII. Solutions of the cases of *right*-angled spherical triangles.

Given.	*Required.*	*Solution.*	
Hypothen. and an angle.	side op. giv. ang.	1. $\sin x = \sin h \,.\, \sin a.$	
	side adj. giv. ang.	2. $\tan x = \tan h \,.\, \cos a.$	
	the other angle.	3. $\cot x = \cos h \,.\, \tan a.$	
Hypothen. and a side.	the other side.	4. $\cos x = \dfrac{\cos h}{\cos s}.$	
	ang. adj. giv. side.	5. $\cos x = \tan s \,.\, \cot h.$	
	ang. op. giv. side.	6. $\sin x = \dfrac{\sin s}{\sin h}.$	
A side and the angle opposite.	the hypothen.	7. $\sin x = \dfrac{\sin s}{\sin a}$	the ambiguous cases.
	the other side.	8. $\sin x = \tan s \,.\, \cot a$	
	the other angle.	9. $\sin x = \dfrac{\cos a}{\cos s}$	
A side and the angle adjacent.	the hypothen.	10. $\cot x = \cos a \,.\, \cot s.$	
	the other side.	11. $\tan x = \tan a \,.\, \sin s.$	
	the other angle.	12. $\cos x = \sin a \,.\, \cos s.$	
The two sides.	the hypothen.	13. $\cos x =$ rectang. cos of the given sides.	
	an angle.	14. $\cot x =$ sin adj. side $\times$ cot op. side.	
The two angles.	the hypothen.	15. $\cos x =$ rectang. cot of the giv. angles	
	a side.	16. $\cos x = \dfrac{\text{cos opp. ang.}}{\text{sin adj. ang.}}.$	

In these formulæ, x denotes the quantity sought.

$a =$ the *given* angle.

$s =$ the *given* side.

$h =$ the hypothenuse.

IX. Solutions of the cases of *oblique*-angled spherical triangles.

GIVEN, *Two sides and an angle opposite one of them.*

Required, 1°. The angle opposite the other given side.

$$\sin x = \frac{\text{sin side op. ang. sought} \times \text{sin giv. ang.}}{\text{sin side oppos. given angle}}.$$

Required, 2°. The angle included between the given sides.

$$\begin{aligned} \cot a' &= \text{tan giv. ang.} \times \text{cos adj. side}, \\ \cos a'' &= \frac{\cos a' \times \text{tan side adj. giv. ang.}}{\text{tan side op. given angle}}, \\ x &= (a' \pm a''). \end{aligned}$$

Required, 3°. The third side.

$$\begin{aligned} \tan a' &= \text{cos giv. ang.} \times \text{tan adj. side}, \\ \cos a'' &= \frac{\cos a' \times \text{cos side op. giv. ang.}}{\text{cos side adj. given angle}}, \\ x &= (a' \pm a''). \end{aligned}$$

In these formulæ, x denotes the quantity sought: a' and a'' are auxiliary angles introduced for the purpose of facilitating the computations.

The angle sought in formula 1 is, in certain cases, ambiguous. In the formulæ 2 and 3, when the angles opposite to the given sides are of the *same species*, we must take the *upper* sign; on the contrary, the *lower* sign. The whole of these formulæ therefore are, in certain cases, ambiguous.

[*continued.*

IX. *continued.* Solutions of the cases of *oblique*-angled spherical triangles.

GIVEN, *Two angles and a side opposite one of them.*

Required, 4°. The side opposite the other given angle.

$$\sin x = \frac{\text{sin ang. op. side sought} \times \text{sin giv. side}}{\text{sin ang. op. given side}}.$$

Required, 5°. The side included between the given angles.

$$\tan a' = \text{tan giv. side} \times \text{cos ang. adj. giv. side},$$

$$\sin a'' = \frac{\sin a' \times \text{tan ang. adj. giv. side}}{\text{tan ang. op. given side}},$$

$$x = (a' \pm a'').$$

Required, 6°. The third angle.

$$\cot a' = \text{cos given side} \times \text{tan adj. angle},$$

$$\sin a'' = \frac{\sin a' \times \text{cos ang. op. giv. side}}{\text{cos ang. adj. given side}},$$

$$x = (a' \pm a'').$$

In these formulæ, x denotes the quantity sought: a' and a'' are auxiliary angles introduced for the purpose of facilitating the computations.

The side sought in formula 4 is, in certain cases, ambiguous. In the formulæ 5 and 6, when the sides opposite the given angles are of the *same species*, we must take the *upper* sign; on the contrary, the *lower* sign. The whole of these formulæ therefore are, in certain cases, ambiguous.

[*continued.*

IX. *continued.* Solutions of the cases of *oblique*-angled spherical triangles.

GIVEN, *Two sides and the included angle.*

Required, 7°. One of the other angles.

$$\tan a' = \cos \text{given angle} \times \tan \textit{given} \text{ side,}$$
$$a'' = \text{the } \textit{base} - a'$$
$$\tan x = \tan \text{given angle} \times \frac{\sin a'}{\sin a''}.$$

In this formula, the *given* side is assumed to be the side opposite the angle sought: the other known side is called the *base.*

Required, 8°. The third side.

$$\tan a' = \cos \text{given angle} \times \tan \textit{given} \text{ side,}$$
$$a'' = \text{the } \textit{base} \sim a',$$
$$\cos x = \cos \textit{given} \text{ side} \times \frac{\cos a''}{\cos a'}.$$

In this formula, either of the given sides may be assumed as the *base;* and the other as the *given* side.

In these formulæ, x denotes the quantity sought: a' and a'' are auxiliary angles introduced for the purpose of facilitating the computations.

If the side sought in formula 8 be small, the formula may not give the value to a sufficient degree of accuracy; and some other mode must be adopted for obtaining the correct value.

[*continued.*

IX. *continued.* Solutions of the cases of *oblique*-angled spherical triangles.

GIVEN, *A side and the two adjacent angles.*

Required, 9°. One of the other sides.

$$\cot a' = \tan \textit{given} \text{ angle} \times \cos \text{given side},$$

$$a'' = \text{the } \textit{vertical} \text{ angle} \sim a',$$

$$\tan x = \tan \text{given side} \times \frac{\cos a'}{\cos a''}.$$

In this formula, the angle, opposite the side sought, is assumed as the *given* angle: the other known angle is called the *vertical* angle.

Required, 10°. The third angle.

$$\cot a' = \tan \textit{given} \text{ angle} \times \cos \text{given side},$$

$$a'' = \text{the } \textit{vertical} \text{ angle} - a',$$

$$\cos x = \cos \textit{given} \text{ angle} \times \frac{\sin a''}{\sin a'}.$$

In this formula, either of the given angles may be assumed as the *vertical* angle; and the other as the *given* angle.

In these formulæ, x denotes the quantity sought: a' and a'' are auxiliary angles introduced for the purpose of facilitating the computations.

If the angle sought in formula 10 be small, the formula may not give the value to a sufficient degree of accuracy; and some other mode must be adopted for obtaining the correct value

[*continued.*

IX. *continued.* Solutions of the cases of *oblique*-angled spherical triangles.

GIVEN, *The three sides.*

Required, 11°. An angle.

$$\sin^2 \tfrac{1}{2} x = \frac{\sin\left(\frac{A+B+C}{2} - B\right) \times \sin\left(\frac{A+B+C}{2} - C\right)}{\sin B \,.\, \sin C},$$

$$\cos^2 \tfrac{1}{2} x = \frac{\sin\left(\frac{A+B+C}{2}\right) \times \sin\left(\frac{A+B+C}{2} - A\right)}{\sin B \,.\, \sin C}.$$

In these formulæ, A, B, C are the three sides of the triangle; and A is assumed as the side opposite to the angle required.

GIVEN, *The three angles.*

Required, 12°. A side.

$$\sin^2 \tfrac{1}{2} x = -\frac{\cos\left(\frac{a+b+c}{2}\right) \times \cos\left(\frac{a+b+c}{2} \sim a\right)}{\sin b \,.\, \sin c},$$

$$\cos^2 \tfrac{1}{2} x = \frac{\cos\left(\frac{a+b+c}{2} \sim b\right) \times \cos\left(\frac{a+b+c}{2} \sim c\right)}{\sin b \,.\, \sin c}.$$

In these formulæ, a, o, c are the three angles of the triangle; and a is assumed as the angle opposite to the side required.

In these formulæ, x denotes the quantity sought. The formulæ, which are resolved by the *cosine*, are used only when the angle or side x is small.

X. Trigonometrical series.

1. $\sin x = x - \frac{x^3}{2\,.\,3} + \frac{x^5}{2\,.\,3\,.\,4\,.\,5} - \&c.$

2. $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{2\,.\,3\,.\,4} - \frac{x^6}{2\,.\,3\,.\,4\,.\,5\,.\,6} + \&c$

3. $\tan x = x + \frac{x^3}{3} + \frac{2\,x^5}{3\,.\,5} + \frac{17\,x^7}{3^2\,.\,5\,.\,7} + \&c.$

4. $\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{3^2\,.\,5} - \frac{2\,x^5}{3^3\,.\,5\,.\,7} - \&c.$

5. $\text{ver-sin}\, x = \frac{x^2}{2} - \frac{x^4}{2\,.\,3\,.\,4} + \frac{x^6}{2\,.\,3\,.\,4\,.\,5\,.\,6} - \&c.$

6. $x = \sin x + \frac{\sin^3 x}{2\,.\,3} + \frac{1\,.\,3 \sin^5 x}{2\,.\,4\,.\,5} + \&c.$

7. $x = \frac{\pi}{2} - \cos x - \frac{\cos^3 x}{2\,.\,3} - \frac{1\,.\,3 \cos^5 x}{2\,.\,4\,.\,5} - \&c.$

8. $x = \tan x - \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x - \&c.$

In the series No. 7, π denotes the periphery of the circle, or 3.14159265.

XI. Multiple arcs.

$$\sin 0 = 0,$$

$$\sin x = \sin x,$$

$$\sin 2x = 2 \sin x \,.\, \cos x,$$

$$\sin 3x = 2 \sin x \,.\, \cos 2x + \sin x,$$

$$\sin 4x = 2 \sin x \,.\, \cos 3x + \sin 2x,$$

&c. &c. &c.

$$\cos 0 = 1,$$

$$\cos x = \cos x,$$

$$\cos 2x = 2 \cos x \,.\, \cos x - 1,$$

$$\cos 3x = 2 \cos x \,.\, \cos 2x - \cos x,$$

$$\cos 4x = 2 \cos x \,.\, \cos 3x - \cos 2x$$

&c. &c. &c.

$$\tan x = \tan x,$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x},$$

$$\tan 3x = \frac{\tan x + \tan 2x}{1 - \tan x \,.\, \tan 2x}$$

$$\tan 4x = \frac{\tan x + \tan 3x}{1 - \tan x \,.\, \tan 3x},$$

&c. &c. &c.